THE QUARTERLY JOURNAL OF MATHEMATICS

OXFORD SECOND SERIES

Volume 1 *No.* 4 *December* 1950

OXFORD

AT THE CLARENDON PRESS

Price 10*s*. 6*d*. *net*

PRINTED IN GREAT BRITAIN BY CHARLES BATEY AT THE UNIVERSITY PRESS OXFORD

THE QUARTERLY JOURNAL OF MATHEMATICS

OXFORD SECOND SERIES

Edited by T. W. CHAUNDY, U. S. HASLAM-JONES, J. H. C. THOMPSON

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THE MINIMUM OF AN INDEFINITE BINARY QUADRATIC FORM

By C. S. DAVIS (*Bristol*)

[Received 12 July 1949]

It is well known* that if

$$f(x,y) = ax^2+bxy+cy^2$$

is an indefinite binary quadratic form with real coefficients and discriminant $d = b^2-4ac > 0$, then integers x, $y \neq 0$, 0 exist such that

$$|f(x,y)| \leqslant \sqrt{(\tfrac{1}{5}d)}.$$

Further, this result is 'best possible', the equality sign being required only for forms equivalent to

$$f_0(x,y) = a(x^2-xy-y^2).$$

Excluding such forms, it is known* that there exist integers x, $y \neq 0$, 0 such that

$$|f(x,y)| \leqslant \sqrt{(\tfrac{1}{8}d)},$$

with strict inequality unless f is equivalent to

$$f_1(x,y) = a(x^2-2xy-y^2).$$

We may express these results in a somewhat different way. If m is the lower bound of $|f(x,y)|$ for integers x, $y \neq 0$, 0 and $k = d/m^2$, the results given above are equivalent to stating that there are no forms $f(x,y)$ with $0 < k < 5$ or $5 < k < 8$, and that $k = 5$, 8 only in the cases noted.

In this note I give a simple proof of these results, together with the additional fact that there are no forms with $12 < k < 13$.

Given $\epsilon_0 > 0$, there exist integers x, $y \neq 0$, 0 such that

$$|f(x,y)| = \frac{m}{1-\epsilon},$$

where $0 \leqslant \epsilon < \epsilon_0$. By an integral unimodular substitution we may suppose

$$f(x,y) = \pm\frac{m}{1-\epsilon}(x-\alpha_1 y)(x-\alpha_2 y),$$

* See, for example, L. E. Dickson, *Introduction to the theory of numbers* (Chicago, 1929), 175–8. Simple proofs of the first two minima have been given recently by G. Pall, *Math. Mag.* 21 (1948), 255, and J. W. S. Cassels, *Annals of Math.* 50 (1949), 676–85 (Lemma 1); and a geometrical proof by K. Ollerenshaw, *J. of London Math. Soc.* 23 (1948), 148–53.

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where α_1, α_2 are real, and hence

$$|(x-\alpha_1 y)(x-\alpha_2 y)| \geqslant 1-\epsilon$$

for all integers $x, y \neq 0, 0$. In particular,

$$|(x-\alpha_1)(x-\alpha_2)| \geqslant 1-\epsilon = \delta, \quad \text{say}, \tag{1}$$

for all integers x.

Write $$\phi(x) = (x-\alpha_1)(x-\alpha_2);$$
its discriminant is $$(1-\epsilon)^2 d/m^2 = \delta^2 k.$$

Let β_1, β_2 ($\beta_1 < \beta_2$) be the roots of $\phi(x) = \delta$, and γ_1, γ_2 be the roots of $\phi(x) = -\delta$, where $\beta_1 < \gamma_1 < \gamma_2 < \beta_2$ if γ_1, γ_2 are real. Then (1) is false if x satisfies $\beta_1 < x < \beta_2$ but not $\gamma_1 < x < \gamma_2$.

We easily find that

$$\beta_2-\beta_1 = \sqrt{(\delta^2 k+4\delta)}, \qquad \gamma_2-\gamma_1 = \sqrt{(\delta^2 k-4\delta)}.$$

Now, if $0 < k < 5$ and δ is sufficiently near to 1 (that is, for sufficiently small ϵ_0), we have $\beta_2-\beta_1 > 2$, and γ_1, γ_2 are imaginary or $\gamma_2-\gamma_1 < 1$. Hence the interval (β_1, β_2) contains at least two integers, while the interval (γ_1, γ_2) contains at most one integer. That is, there exists an integer ξ which satisfies $\beta_1 < \xi < \beta_2$ but not $\gamma_1 < \xi < \gamma_2$, and so $|\phi(\xi)| < \delta$, contrary to (1). Hence we cannot have $0 < k < 5$.

If $k = 5$, we still have $\beta_2-\beta_1 > 2$, and now

$$\gamma_2-\gamma_1 = \sqrt{(5\delta^2-4\delta)} \leqslant \sqrt{\delta} \leqslant 1.$$

Here again $\delta < 1$ gives $\gamma_2-\gamma_1 < 1$ and so must be excluded. Hence $\delta = 1$ and $\gamma_2-\gamma_1 = 1$. The closed interval (γ_1, γ_2) can then contain two integers only if these are the end points. We may suppose without loss of generality that $\gamma_1 = 0$, and this gives $\phi(x) = x^2-x-1$ and so

$$f \sim a(x^2-xy-y^2) = f_0.$$

Since f_0 has the minimum a and discriminant $5a^2$, we actually have $k = 5$ for $f \sim f_0$.

Similarly, if ϵ_0 is sufficiently small, $5 < k < 8$ gives $\beta_2-\beta_1 > 3$, $\gamma_2-\gamma_1 < 2$ and so, as before, these values of k are impossible. Again, if $k = 8$, γ_1 must be an integer, say zero, leading to $f \sim f_1$.

In the same way we may exclude the further range $12 < k < 13$, since such values of k lead to $\beta_2-\beta_1 > 4$ and $\gamma_2-\gamma_1 < 3$. In addition, we find that $k = 13$ if and only if $f \sim a(x^2-3xy-y^2)$.

AN OSCILLATION THEOREM OF TAUBERIAN TYPE

By N. A. BOWEN *and* A. J. MACINTYRE (*Aberdeen*)

[Received 27 July 1949]

1. In this paper we answer, for the Tauberian theorem of Valiron* and Titchmarsh,† a question which might well be raised for any Tauberian theorem. If the existence of $\lim_{x\to\infty} A(x,x_n)$ implies that of $\lim_{n\to\infty} B(x_n)$, does it follow that

$$\limsup_{n\to\infty} B(x_n) - \liminf_{n\to\infty} B(x_n)$$

will be arbitrarily small whenever

$$\limsup_{x\to\infty} A(x,x_n) - \liminf_{x\to\infty} A(x,x_n)$$

is sufficiently small? If the answer is in the affirmative, the Tauberian theorem appears as a special case of a more general theorem.

This is the case (trivially) for Tauber's theorem.‡ If $a_n = o(1/n)$, then

$$\liminf_{x\to 1-0} \sum_0^\infty a_n x^n = \liminf_{n\to\infty} \sum_0^n a_k \leqslant \limsup_{n\to\infty} \sum_0^n a_k = \limsup_{x\to 1-0} \sum_0^\infty a_n x^n.$$

Generalizations of this result when the condition $a_n = o(1/n)$ is relaxed to $a_n = O(1/n)$ or $\sum_1^n ka_k = O(n)$ have been given recently,§ but these do not contain, for example, Littlewood's Tauberian theorem‖ as a limiting case.

The theorem we prove may be stated as follows.

THEOREM. *If* $f(z) = \prod_{n=1}^\infty (1+z/a_n)$ *is an integral function of order* ρ *with* $0 < \rho < 1$ *whose zeros are all negative and if*

$$\begin{aligned} &\liminf_{x\to\infty} x^{-\rho}\log f(x) \geqslant \lambda, \\ &\limsup_{x\to\infty} x^{-\rho}\log f(x) \leqslant \Lambda \quad (\Lambda > \lambda) \end{aligned} \tag{1}$$

are positive and finite, then

$$\phi(\lambda,\Lambda) \leqslant \liminf_{n\to\infty} n^{-1/\rho} a_n \leqslant \limsup_{n\to\infty} n^{-1/\rho} a_n \leqslant \Phi(\lambda,\Lambda), \tag{2}$$

where $\phi(\lambda,\Lambda)$ *and* $\Phi(\lambda,\Lambda)$ *are positive and* $\Phi(\lambda,\Lambda)/\phi(\lambda,\Lambda)$ *is arbitrarily near unity provided that* Λ/λ *is sufficiently near unity.*

* (1), 237–43 or (2), 121–7. † (3). ‡ Given, e.g., in (4), 10.
§ (5), (6), (7), (8), (9), (10). ‖ Given, e.g., in (4), 233.

The existence of $\phi(\lambda, \Lambda)$ and $\Phi(\lambda, \Lambda)$ is elementary. Our proof of their limiting behaviour is similar to that given by Bowen* for the case $\lambda = \Lambda$, which is the theorem of Valiron and Titchmarsh. We base the argument on Nevanlinna's 'two-constant' inequality† in place of the theorem of Montel. Our proof does not yield the best values of ϕ and Φ. We make, however, a conjecture on this question.

2. We use the following form of the 'two-constant' inequality.

LEMMA. *If $u(z)$ is regular in the region*

$$\tfrac{1}{2}r \leqslant |z| \leqslant 2r, \quad 0 \leqslant \arg z \leqslant \omega < \pi,$$

and satisfies $|u(z)| \leqslant m$ for $\tfrac{1}{2}r \leqslant z \leqslant 2r$ and $|u(z)| \leqslant M$ $(> m)$ on the rest of the boundary of the region, then, if $0 < \theta < \omega$,

$$|u(re^{i\theta})| \leqslant m^{\mu} M^{1-\mu} \tag{3}$$

where $\mu > 0$ depends only on θ.

This lemma goes beyond the usual statement of the general inequality only in taking account of elementary properties of the harmonic measure in the special case under consideration. Proof is omitted.

3. In proving the theorem we may evidently take any prescribed value for λ. Only the ratio Λ/λ is significant. [Consider $f(\alpha z)$ in place of $f(z)$.] In conditions of the form $|z| > r_0$ the value of r_0 will depend on the particular $f(z)$ involved. The other conditional inequalities we use will be independent of $f(z)$. It will be convenient to replace (1) by

$$\pi \operatorname{cosec} \pi\rho \leqslant \liminf_{x\to\infty} x^{-\rho} \log f(x) \leqslant \limsup_{x\to\infty} x^{-\rho} \log f(x)$$
$$\leqslant \pi(1+h) \operatorname{cosec} \pi\rho, \tag{4}$$

and to prove that, if $\epsilon > 0$ is given and h is sufficiently small, then

$$1-\epsilon \leqslant \liminf_{r\to\infty} r^{-\rho} n(r) \leqslant \limsup_{r\to\infty} r^{-\rho} n(r) \leqslant 1+\epsilon, \tag{5}$$

where $n(r)$ is the number of zeros $-a_n$ of $f(z)$ in the range $0 < a_n < r$.

We have, for $|\arg z| < \pi$,

$$\log f(z) = \int_0^\infty \frac{z n(t)\, dt}{t(z+t)}. \tag{6}$$

* (**11**). A similar proof appeared at the same time in (**12**).

† Stated in (**13**), 41.

When z is restricted by $|\arg z| \leqslant \pi-\frac{1}{2}\delta$, $\delta > 0$, then

$$|z+t|^2 \geqslant |z|^2+t^2-2|z|t\cos\tfrac{1}{2}\delta = (|z|+t)^2-2|z|t(1+\cos\tfrac{1}{2}\delta)$$
$$= (|z|+t)^2-4|z|t\cos^2(\tfrac{1}{4}\delta) \geqslant (|z|+t)^2\sin^2(\tfrac{1}{4}\delta)$$

and
$$|\log f(z)| \leqslant \int\limits_0^\infty \frac{|z|n(t)\,dt}{t|z+t|} \leqslant \operatorname{cosec}\tfrac{1}{4}\delta\log f|z|. \tag{7}$$

For each $\epsilon_1 > 0$ we thus have

$$|\log f(z)| < \pi(1+h+\epsilon_1)|z|^\rho \operatorname{cosec}\pi\rho\operatorname{cosec}\tfrac{1}{4}\delta \tag{8}$$

from (4), when $|z|$ is sufficiently large.

We now apply the lemma to

$$\phi(z) = z^{-\rho}\log f(z)-\pi\operatorname{cosec}\pi\rho, \tag{9}$$

with $\omega = \pi-\frac{1}{2}\delta$ and $\theta = \pi-\delta$. Provided that r is sufficiently large, we can take

$$M = \{(1+2h)\operatorname{cosec}\tfrac{1}{4}\delta+1\}\pi\operatorname{cosec}\pi\rho, \qquad m = 2h\pi\operatorname{cosec}\pi\rho,$$

since we are assuming (4). We deduce that, provided that h is sufficiently small $[h < h(\epsilon_1, \delta)]$,

$$|z^{-\rho}\log f(z)-\pi\operatorname{cosec}\pi\rho| < \epsilon_1 \tag{10}$$

where $z = re^{i(\pi-\delta)}$ and r is sufficiently large. From (10) we evidently have

$$|\mathscr{I}\{\log f(re^{i(\pi-\delta)})\}-\pi r^\rho\operatorname{cosec}\pi\rho\sin\rho(\pi-\delta)| < \epsilon_1 r^\rho \tag{11}$$

for all large r.

The proof is completed by showing that $\mathscr{I}\{\log f(re^{i(\pi-\delta)})\}$ is, for small δ, so near to $\pi n(r)$ that our theorem follows from (11). Taking the imaginary part of (6) with $z = r\exp i(\pi-\delta)$ we have

$$\mathscr{I}\{\log f(re^{i(\pi-\delta)})\} = \int\limits_0^\infty \frac{rn(t)\sin\delta\,dt}{(t-r\cos\delta)^2+r^2\sin^2\delta}. \tag{12}$$

Now, if $\qquad t < r(\cos\delta-\delta^{\frac{1}{4}})/(1+\delta^{\frac{1}{4}}) = \lambda_1 r,$

we have $\qquad r\cos\delta-t > \delta^{\frac{1}{4}}(r+t)$;

and similarly, if $\qquad t > r(\cos\delta+\delta^{\frac{1}{4}})/(1-\delta^{\frac{1}{4}}) = \lambda_2 r,$

we have $\qquad t-r\cos\delta > \delta^{\frac{1}{4}}(r+t).$

In each of these ranges of t the kernel of the integral (12) satisfies

$$\frac{r\sin\delta}{(t-r\cos\delta)^2+r^2\sin^2\delta} < \frac{r\sin\delta}{\delta^{\frac{1}{2}}(r+t)^2} < \frac{\delta^{\frac{1}{2}}r}{t(r+t)}. \tag{13}$$

Hence

$$0 \leqslant \mathscr{I}\{\log f(re^{i(\pi-\delta)})\} - \int\limits_{\lambda_1 r}^{\lambda_2 r} \frac{rn(t)\sin\delta\,dt}{(t-r\cos\delta)^2+r^2\sin^2\delta} < \delta^{\frac{1}{2}}\log f(r). \tag{14}$$

Now the integral appearing in this inequality clearly lies between

$$n(\lambda_1 r)\int\limits_{\lambda_1 r}^{\lambda_2 r} \frac{r\sin\delta\,dt}{(t-r\cos\delta)^2+r^2\sin^2\delta}$$

$$= n(\lambda_1 r)\left\{\tan^{-1}\left(\frac{\lambda_2 r-r\cos\delta}{r\sin\delta}\right)-\tan^{-1}\left(\frac{\lambda_1 r-r\cos\delta}{r\sin\delta}\right)\right\}$$

$$= n(\lambda_1 r)\left\{\tan^{-1}\left(\frac{\delta^{\frac{1}{2}}(1+\cos\delta)}{\sin\delta(1-\delta^{\frac{1}{2}})}\right)+\tan^{-1}\left(\frac{\delta^{\frac{1}{2}}(1+\cos\delta)}{\sin\delta(1+\delta^{\frac{1}{2}})}\right)\right\} = \psi(\delta)n(\lambda_1 r) \tag{15}$$

and

$$\psi(\delta)n(\lambda_2 r). \tag{16}$$

By choice of δ sufficiently small we make λ_1 and λ_2 arbitrarily near unity, $\psi(\delta)$ and $\pi\operatorname{cosec}\pi\rho\sin\rho(\pi-\delta)$ arbitrarily near π. If we impose the over-riding restriction $h < 1$, then, by (4), $\delta^{\frac{1}{2}}r^{-\rho}\log f(r)$ is arbitrarily small when δ is sufficiently small for all sufficiently large r. Since δ is fixed, ϵ_1 can still be chosen as small as we please, and (11) will be satisfied for large r if h is sufficiently small. This proves the theorem.

4. A conjecture as to the best values of ϕ and Φ in (2) may be reached as follows. We wish to know the greatest possible oscillation of $r^{-\rho}n(r)$ when the oscillation of $x^{-\rho}\log f(x)$ is given. If, however, the extreme limits of $r^{-\rho}n(r)$ were given, we should expect those of $x^{-\rho}\log f(x)$ to be as close as possible when $r^{-\rho}n(r)$ oscillates as rapidly as possible. This suggests that ϕ and Φ may be determined as follows.

Given $L > l > 0$, define the non-decreasing function $\nu(r)$ for $r > 0$ by means of the double sequence r_n $(n = 0, \pm 1, \pm 2, \ldots)$, where $r_0 = 1$, $r_{n+1} = (L/l)^{1/\rho}r_n$ as

$$\nu(r) = Lr_n^\rho \quad (r_n \leqslant r < r_{n+1}).$$

Clearly $$\liminf_{r\to\infty} r^{-\rho}\nu(r) = l, \qquad \limsup_{r\to\infty} r^{-\rho}\nu(r) = L.$$

Define $\mu(r)$ by $$\mu(r) = \int\limits_0^\infty \frac{r\nu(t)\,dt}{t(r+t)}.$$

Then $\mu(r)$ will oscillate between two values λ and Λ $(> \lambda)$. The conjecture is that l and L are determined uniquely by λ and Λ, and are the best values of ϕ and Φ.

5. The enunciation given by Valiron* covered functions of non-integral order not restricted to lie between 0 and 1. It is clear that our theorem may be similarly generalized. It is only necessary to replace the appeal to Montel's theorem made in (**11**) 97 by the Nevanlinna 'two-constant' inequality stated in § 2 above.†

* (**1**), 237 or (**2**), 121.

† The following corrections to this paper (**11**) may be noted: in (29) omit $(-1)^p$; on p. 98 line 1 for $\pi/(p+1)$ read $\pi/2(p+1)$; in (38) omit r^ρ (twice); two lines below (38) for (31) read (29).

REFERENCES

1. G. Valiron, *Annales de Toulouse* (3), 5 (1914), 117–257.
2. —— Thesis (Toulouse, 1914).
3. E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 26 (1926), 185–200.
4. —— *Theory of Functions* (Oxford, 1939).
5. H. Hadwiger, *Comment. Math. Helvetici* 16 (1944), 209–14.
6. R. P. Agnew, *Duke Math. Journal* 12 (1945), 27–36.
7. H. Hadwiger, *Comment. Math. Helvetici* 20 (1947), 319–32.
8. —— *Revista Mat. Hispano-Americana* (4), 7 (1947), 65–9.
9. P. Hartman, *American J. of Math.* 69 (1947), 599–606.
10. A. Wintner, *Comment. Math. Helvetici* 20 (1947), 216–22.
11. N. A. Bowen, *Quart. J. of Math.* (Oxford), 19 (1948), 90–100.
12. M. Heins, *Annals of Math.* (1), 49 (1948), 200–13.
13. R. Nevanlinna, *Eindeutige Analytische Funktionen* (Berlin, 1936).

ON THE SECOND VARIATION OF THE VOLUME INTEGRAL WHEN THE BOUNDARY IS VARIABLE

By E. T. DAVIES (*Southampton*)

[Received 2 September 1949]

THIS note is a sequel to a previous note (**1**) in which the boundary was assumed to be fixed. I shall use the same notation.

Let us assume that we have a minimal subspace $\underset{0}{V_m}$ given parametrically by

$$x^\lambda = x^\lambda(u^1, u^2,..., u^m). \tag{1}$$

A subspace V_m neighbouring $\underset{0}{V_m}$ will be called *admissible* if its boundary points are given by

$$u^i = u^i(y^1, y^2,..., y^r), \qquad x^\lambda = x^\lambda(y^1,..., y^r) \quad (m \leqslant r \leqslant n) \tag{2}$$

for values of the parameters (y) near (0), where we assume that the functions in (2) are of class C^2 and that for $(y) = (0)$ they give the boundary points of $\underset{0}{V_m}$. An admissible r-parameter family of subspaces is therefore obtained if we set

$$x^\lambda = \phi^\lambda(u^1,..., u^m; y^1,..., y^r), \tag{3}$$

where ϕ^λ (i) are of class C^2, (ii) take the values (1) for $(y) = (0)$, and (iii) take the values (2) on the boundary.

Now let $y^\alpha(t)$ $(\alpha = 1, 2,..., r)$ be a set of functions of class C^2 for t near $t = 0$, with $y^\alpha(0) = 0$, and put

$$\phi^\lambda\{u; y(t)\} = x^\lambda(u; t) \tag{4}$$

so that we have in
$$x^\lambda = x^\lambda(u^1,..., u^m; t) \tag{5}$$

a one-parameter family of admissible subspaces satisfying, at the boundary, the identity

$$x^\lambda\{y(t)\} \equiv x^\lambda[u\{y(t)\}, t]. \tag{6}$$

We shall need the following relation satisfied by the derivatives at the boundary, on putting $du^i/dt = z^i$:

$$\frac{dx^\lambda}{dt} = \frac{\partial x^\lambda}{\partial t} + \frac{\partial x^\lambda}{\partial u^i}\frac{du^i}{dt} = v^\lambda + B^\lambda_i z^i. \tag{7}$$

We consider therefore a one-parameter family of admissible V_m's, each member of which is bounded by a V_{m-1} of parametric equations

$$x^\lambda = x^\lambda(\rho^1, \rho^2, \dots, \rho^{m-1}),$$

the points of each V_{m-1} moving on a V_r of parametric equations

$$x^\lambda = x^\lambda(y^1, y^2, \dots, y^r),$$

so that, along the V_r, on writing $\pi^\alpha = dy^\alpha/dt$, we shall have

$$\frac{dx^\lambda}{dt} = \frac{\partial x^\lambda}{\partial y^\alpha}\pi^\alpha, \tag{8}$$

$$\frac{d^2x^\lambda}{dt^2} = \frac{\partial^2 x^\lambda}{\partial y^\alpha \partial y^\beta}\pi^\alpha\pi^\beta + \frac{\partial x^\lambda}{\partial y^\alpha}\frac{d^2y^\alpha}{dt^2}. \tag{9}$$

If the integral

$$L(t) = \int\limits_\Sigma F\left(x, \frac{\partial x}{\partial u}\right) du \quad \text{where} \quad du = du^1 du^2 \dots du^m,$$

is taken over an admissible V_m of the kind described above, the x, the $\partial x/\partial u$, and the Σ all depend on t. The first and second variations of integrals of this type, involving derivatives of arbitrary order, are given by de Donder (**4**). On denoting by Γ the boundary of Σ, and writing

$$(-1)^{i+1}p_i = \frac{\partial(u^1, u^2, \dots, u^{i-1}, u^{i+1}, \dots, u^m)}{\partial(\rho^1, \rho^2, \dots, \rho^{m-1})},$$

the formula (76) of de Donder can be written

$$\frac{dL}{dt} = \int\limits_\Sigma \frac{\partial F}{\partial t} du + \int\limits_\Gamma p_i(FZ^i)\, d\rho, \tag{10}$$

which does not coincide with a corresponding result given by Tucker [(**2**) (6.3)]. Similarly the formula (172) of de Donder becomes

$$\frac{d^2L}{dt^2} = \int\limits_\Sigma \frac{\partial^2 F}{\partial t^2} du + \int\limits_\Gamma p_i\left[\left(\frac{\partial F}{\partial t} Z^i\right) + \frac{d}{dt}(FZ^i) + FZ^i\frac{dZ^k}{du^k} - FZ^k\frac{dZ^i}{du^k}\right] d\rho, \tag{11}$$

which I shall proceed to express in another form.

We first note that the expression

$$\frac{\partial F}{\partial t} = F_\lambda|v^\lambda + F|^i_\lambda \partial_i v^\lambda = F|^i_\lambda D_i v^\lambda \tag{12}$$

can be modified, on remarking that $F_\lambda| - \dfrac{d}{du^i}(F|^i_\lambda)$ is the mean-curvature vector M_λ of V_m in V_n, to

$$\frac{\partial F}{\partial t} = M_\lambda v^\lambda + \frac{d}{du^i}(F|^i_\lambda v^\lambda). \tag{13}$$

Differentiating (12) and writing

$$\Omega = F_{\lambda\mu}|v^\lambda v^\mu + 2F_\lambda|^j_\mu v^\lambda \partial_j v^\mu + F|^{ij}_{\lambda\mu}\partial_i v^\lambda \partial_j v^\mu, \tag{14}$$

we get

$$\frac{\partial^2 F}{\partial t^2} = \Omega + F_\lambda|\frac{\partial v^\lambda}{\partial t} + F|^i_\lambda \frac{d}{du^i}\left(\frac{\partial v^\lambda}{\partial t}\right) = \Omega + M_\lambda \frac{\partial v^\lambda}{\partial t} + \frac{d}{du^i}\left(F|^i_\lambda \frac{\partial v^\lambda}{\partial t}\right). \tag{15}$$

If we now use (5) and (6) of (1), we can deduce

$$F_\lambda|^j_\mu = F|^{ij}_{\rho\mu}\Gamma^\rho_{\lambda\nu} B^\nu_i + F|^j_\rho \Gamma^\rho_{\lambda\mu},$$

$$F_{\lambda\mu}| = F|^{ij}_{\rho\sigma}\Gamma^\sigma_{\mu\tau} B^\tau_j \Gamma^\rho_{\lambda\nu} B^\nu_i + F|^i_\rho \Gamma^\rho_{\mu\tau}\Gamma^\tau_{\lambda\nu} B^\nu_i + F|^i_\rho \partial_\mu \Gamma^\rho_{\lambda\nu} B^\nu_i$$

and rewrite (14) as

$$\Omega = F|^{ij}_{\lambda\mu} D_i v^\lambda D_j v^\mu + B^\rho_\sigma R^{\cdot\cdot\cdot\sigma}_{\lambda\rho,\mu} v^\lambda v^\mu + \frac{d}{du^i}(F|^i_\lambda \Gamma^\lambda_{\mu\nu} v^\mu v^\nu),$$

so that (15) becomes

$$\frac{\partial^2 F}{\partial t^2} = F|^{ij}_{\lambda\mu} D_i v^\lambda D_j v^\mu + B^\rho_\sigma R^{\cdot\cdot\cdot\sigma}_{\lambda\rho,\mu} v^\lambda v^\mu + M_\lambda \frac{\partial v^\lambda}{\partial t} + \\ + \frac{d}{du^i}\left[F|^i_\lambda\left(\frac{\partial v^\lambda}{\partial t} + \Gamma^\lambda_{\mu\nu} v^\mu v^\nu\right)\right].$$

Finally

$$\frac{d^2 L}{dt^2} = \int_\Sigma \left\{F|^{ij}_{\lambda\mu} D_i v^\lambda D_j v^\mu + B^\rho_\sigma R^{\cdot\cdot\cdot\sigma}_{\lambda\rho,\mu} v^\lambda v^\mu + M_\lambda \frac{\partial v^\lambda}{\partial t}\right\} du + \\ + \int_\Gamma p_i\left\{F|^i_\lambda\left(\frac{\partial v^\lambda}{\partial t} + \Gamma^\lambda_{\mu\nu} v^\mu v^\nu\right) + \frac{\partial F}{\partial t} Z^i + \frac{d}{dt}(FZ^i) + FZ^i \frac{dZ^k}{du^k} - FZ^k \frac{dZ^i}{du^k}\right\} d\rho. \tag{16}$$

This expression for the second variation does not involve any assumption about the first variation.

Returning now to the first variation, we notice that, on using (13) and (1) (6), the right-hand side of (10) may be modified to

$$\frac{dL}{dt} = \int_\Sigma M_\lambda v^\lambda\, du + \int_\Gamma F p_i (B^i_\lambda v^\lambda + Z^i)\, d\rho,$$

and, on using (7), to

$$\frac{dL}{dt} = \int\limits_{\Sigma} M_\lambda v^\lambda \, du + \int\limits_{\Gamma} Fp_i \, B^i_\lambda \frac{dx^\lambda}{dt} \, d\rho. \tag{17}$$

Sufficient conditions for the vanishing of the first variation are therefore

$$\text{(i)} \quad M_\lambda = 0 \text{ throughout } \Sigma, \tag{18}$$

$$\text{(ii)} \quad Fp_i \, B^i_\lambda \frac{dx^\lambda}{dt} = 0 \text{ along } \Gamma.$$

On taking account of (8) the condition (ii) will be satisfied for any variation π^α along V_r provided that

$$Fp_i \, B^i_\lambda \frac{\partial x^\lambda}{\partial y^\alpha} = 0. \tag{19}$$

This demands that the Σ-component of the variation at the boundary Γ shall lie entirely in Γ, i.e. shall be perpendicular to the vector Fp_i specifying the unique normal to Γ in V_m. For our present purpose I shall make the stronger assumption that there is no Σ-component of the variation at all, and that $B^i_\lambda v^\lambda$ and Z^i separately vanish along Γ. From this follows, on using (8) and (9),

$$v^\lambda = \frac{\partial x^\lambda}{\partial t} = \frac{dx^\lambda}{dt} = \frac{\partial x^\lambda}{\partial y^\alpha} \pi^\alpha,$$

$$\frac{\partial v^\lambda}{\partial t} = \frac{d^2 x^\lambda}{dt^2} = \frac{\partial x^\lambda}{\partial y^\alpha} \frac{d^2 y^\alpha}{dt^2} + \frac{\partial^2 x^\lambda}{\partial y^\alpha \partial y^\beta} \pi^\alpha \pi^\beta,$$

and (16), on using (18) and (19), takes the simplified form

$$\frac{d^2 L}{dt^2} = \int\limits_{\Sigma} \{F|^{ij}_{\lambda\mu} \, D_i v^\lambda D_j v^\mu + B^\rho_\sigma \, R^{\cdot\cdot\cdot\sigma}_{\lambda\rho,\mu} \, v^\lambda v^\mu\} \, du +$$

$$+ \int\limits_{\Gamma} Fp_i \, B^i_\lambda \left(\frac{\partial^2 x^\lambda}{\partial y^\alpha \partial y^\beta} + \Gamma^\lambda_{\mu\nu} \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta}\right) \pi^\alpha \pi^\beta \, d\rho.$$

If we now denote by $\Gamma^\gamma_{\alpha\beta}$ the three-index symbols of Christoffel for the metric of V_r as a subspace of V_n, and use (19) and the fact that

$$H^{\cdot\cdot\lambda}_{\alpha\beta} = \frac{\partial^2 x^\lambda}{\partial y^\alpha \partial y^\beta} + \Gamma^\lambda_{\mu\nu} \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} - \Gamma^\gamma_{\alpha\beta} \frac{\partial x^\lambda}{\partial y^\gamma}$$

is the first tensor of Eulerian curvature of V_r in V_n, we shall finally have

$$\frac{d^2 L}{dt^2} = \int\limits_{\Sigma} \{F|^{ij}_{\lambda\mu} \, D_i v^\lambda D_j v^\mu + B^\rho_\sigma \, R^{\cdot\cdot\cdot\sigma}_{\lambda\rho,\mu} \, v^\lambda v^\mu\} \, du + \int\limits_{\Gamma} Fp_i \, B^i_\lambda \, H^{\cdot\cdot\lambda}_{\alpha\beta} \pi^\alpha \pi^\beta \, d\rho. \tag{20}$$

Now let us consider the case of an extremal curve with one end-point fixed and the other end movable upon a hypersurface which is perpendicular to the tangent vector to the curve at their point of intersection. In that case $m = 1$, $r = n-1$ and the integral over Γ becomes the finite term $FB^1_\lambda H^{\cdot\cdot\lambda}_{\alpha\beta}\pi^\alpha\pi^\beta$ evaluated at the free extremity. In that case $FB^1_\lambda = i_\lambda$, the unit vector tangent to the curve. If, therefore, we write $i_\lambda H^{\cdot\cdot\lambda}_{\alpha\beta} = h_{\alpha\beta}$, the finite term in question becomes $(h_{\alpha\beta}\, dy^\alpha dy^\beta)/dt^2$ where $h_{\alpha\beta}\, dy^\alpha dy^\beta$ is the second fundamental form of the hypersurface upon which the end-point moves. This corresponds to a remark made by Morse [(**3**) (3.19)].

REFERENCES

1. E. T. Davies, *Quart. J. of Math.* (Oxford) 13 (1942), 58–64.
2. A. W. Tucker, *Annals of Math.* 35 (1934), 341–50.
3. M. Morse, *Calculus of Variations in the Large*, American Math. Soc., Colloquium Publications, XVIII (1934).
4. T. de Donder, *Théorie Invariantive du Calcul des Variations* (Paris, 1935).

NOTE ON A BINARY QUARTIC FORM

By H. DAVENPORT (*London*)

[Received 4 September 1949]

1. LET x and y be linear forms in two variables u and v, with real coefficients and with determinant 1. We consider the minimum of the binary quartic form $x^2(x^2-y^2)$, when u and v take integral values, not both zero.* The best possible inequality for this minimum is given by the following theorem.

THEOREM. *There exist integers u, v, not both zero, such that*

$$|x^2(x^2-y^2)| \leqslant \frac{1}{1+2\sqrt{5}}. \tag{1}$$

This constant is the least possible, in the sense that there exist special linear forms for which (1) *has no solution with strict inequality.*

The result can also be expressed in the language of the geometry of numbers; it asserts that the critical determinant of the region in the plane defined by $|x^2(x^2-y^2)| \leqslant 1$ is $\sqrt{(1+2\sqrt{5})}$.

2. For the proof of the first part of the theorem we need two lemmas.

LEMMA 1. *Let ϕ be any irrational number. Then either* (*a*) *the inequality*

$$\left|\phi-\frac{u}{v}\right| < \frac{1}{v^2\sqrt{8}} \tag{2}$$

has an infinity of solutions in integers u, v with $v > 0$, or (*b*) *there exist integers α, β, γ, δ, with $\alpha\delta-\beta\gamma = \pm 1$, such that*

$$\frac{\alpha\phi+\beta}{\gamma\phi+\delta} = \theta, \quad \textit{where} \quad \theta = \tfrac{1}{2}(1+\sqrt{5}). \tag{3}$$

This forms part of a series of results on Diophantine approximation, due to Markoff and Hurwitz.†

LEMMA 2. *Let A and B be any real numbers, not both zero, and let θ be as defined in* (3). *Then there exists an integer n* (*positive, negative, or zero*) *such that*

$$|\{A(-\theta^2)^n-1\}\{B(-\theta^2)^n-1\}| \leqslant \frac{5}{1+2\sqrt{5}}. \tag{4}$$

* Some general results concerning the minimum of a binary quartic form are given by Dr. C. S. Davis in a paper in course of publication in *Acta Math.*

† See Koksma, *Diophantische Approximationen*, 31.

Quart. J. Math. Oxford (2) 1 (1950), 253–61.

On the other hand, if A and B are the real numbers defined by

$$A+B = \frac{2}{\theta^4(1+2\sqrt{5})}, \qquad AB = -\frac{2}{\theta^5(1+2\sqrt{5})}, \tag{5}$$

then (4) *is never satisfied with strict inequality, but is satisfied with equality when n is* 0 *or* 1 *or* 2.

Proof. Let

$$\Pi_n = \{A(-\theta^2)^n-1\}\{B(-\theta^2)^n-1\}. \tag{6}$$

Obviously we can suppose without loss of generality that $|A| \geqslant |B|$. As the assertion of the lemma is unaltered in meaning if A and B are replaced by $A(-\theta^2)^k$ and $B(-\theta^2)^k$, where k is any integer, we can further suppose without loss of generality that

$$0 < A \leqslant 1 < A\theta^4, \qquad |B| \leqslant A. \tag{7}$$

Case 1. Suppose that $B\theta^4 \geqslant 1$. Then

$$|\Pi_2| = (A\theta^4-1)(B\theta^4-1), \qquad |\Pi_0| = (1-A)(1-B).$$

By the inequality of the arithmetic and geometric means, we have

$$|\Pi_2| \leqslant (C\theta^4-1)^2, \qquad |\Pi_0| \leqslant (1-C)^2,$$

where $C = \frac{1}{2}(A+B)$. The lesser of the two numbers on the right is greatest when $C\theta^4-1 = 1-C$. Hence one at least of $|\Pi_2|$ and $|\Pi_0|$ does not exceed

$$\left(\frac{\theta^4-1}{\theta^4+1}\right)^2 = \left(\frac{\theta^2-\theta^{-2}}{\theta^2+\theta^{-2}}\right)^2 = \frac{5}{9} < \frac{5}{1+2\sqrt{5}}.$$

Case 2. Suppose that $0 \leqslant B\theta^4 < 1$. Now

$$|\Pi_2| = (A\theta^4-1)(1-B\theta^4), \qquad |\Pi_0| = (1-A)(1-B).$$

Hence
$$|\Pi_2| \leqslant A\theta^4-1, \qquad |\Pi_0| \leqslant 1-A.$$

Plainly one at least of these does not exceed

$$\frac{\theta^4-1}{\theta^4+1} = \frac{\sqrt{5}}{3} < \frac{5}{1+2\sqrt{5}}.$$

Case 3. Suppose that $B < 0$ and that $|B|\theta^2 \geqslant 1$. Now

$$|\Pi_1| = (A\theta^2+1)(|B|\theta^2-1), \qquad |\Pi_0| = (1-A)(1+|B|).$$

Since $|B| \leqslant A$, we have

$$|\Pi_1| \leqslant A^2\theta^4-1, \qquad |\Pi_0| \leqslant 1-A^2.$$

One at least of these does not exceed $(\theta^4-1)/(\theta^4+1)$, as in Case 2.

Case 4. Suppose finally that $B < 0$ and $|B|\theta^2 < 1$. Now

$$|\Pi_2| = (A\theta^4-1)(1+|B|\theta^4),$$
$$|\Pi_1| = (A\theta^2+1)(1-|B|\theta^2),$$
$$|\Pi_0| = (1-A)(1+|B|).$$

Let $P = A|B|$ and $S = A+B = A-|B|$. The three expressions are

$$\left.\begin{aligned} \theta^8 P+\theta^4 S-1, \\ -\theta^4 P+\theta^2 S+1, \\ -P--S+1. \end{aligned}\right\} \tag{8}$$

If these are multiplied by the positive numbers $\theta^4+\theta^2$, $\theta^8-\theta^4$, $\theta^{10}+\theta^8$ and added, the result is $-\theta^4-\theta^2+\theta^8-\theta^4+\theta^{10}+\theta^8$. Hence one at least of the three expressions (8) does not exceed

$$\frac{-\theta^4-\theta^2+\theta^8-\theta^4+\theta^{10}+\theta^8}{\theta^4+\theta^2+\theta^8-\theta^4+\theta^{10}+\theta^8} = \frac{(\theta^4-\theta^{-4})+2(\theta^2-\theta^{-2})}{(\theta^4+\theta^{-4})+2\theta^2}.$$

Since $\qquad \theta^2-\theta^{-2} = \sqrt{5}, \qquad \theta^2+\theta^{-2} = 3, \qquad \theta^4+\theta^{-4} = 7,$

the above number is

$$\frac{5\sqrt{5}}{7+2\theta^2} = \frac{5\sqrt{5}}{10+\sqrt{5}} = \frac{5}{1+2\sqrt{5}}.$$

This completes the proof of the first assertion of Lemma 2.

We now determine A and B so that the three expressions in (8) are equal. From the last two of them, we obtain

$$P(\theta^4-1) = S(\theta^2+1), \quad \text{i.e.} \quad P(\theta^2-1) = S. \tag{9}$$

From the first two,

$$P(\theta^8+\theta^4)+S(\theta^4-\theta^2) = 2.$$

Hence

$$P = \frac{2}{\theta^8+\theta^4+(\theta^4-\theta^2)(\theta^2-1)} = \frac{2}{\theta^5(\theta^3+\theta^{-3}+\theta-\theta^{-1})} = \frac{2}{\theta^5(1+2\sqrt{5})}.$$

This value of P, and the value for S which results from (9), give precisely the definition of the special numbers A and B which was laid down in (5). The numerical values are

$$P = 0{\cdot}0329..., \qquad S = 0{\cdot}0533..., \tag{10}$$

from which one finds

$$A = 0{\cdot}210..., \qquad B = -0{\cdot}156.... \tag{11}$$

These numbers fall under Case 4 above, as they should, and it is plain from the preceding construction that

$$\Pi_0 = \Pi_1 = -\Pi_2 = \frac{5}{1+2\sqrt{5}} \tag{12}$$

for these special numbers A and B.

It remains to be proved that $|\Pi_n|$, for any other value of n, is greater than $5/(1+2\sqrt{5})$. First, if n is even and greater than 2, we have

$$|\Pi_n| = (A\theta^{2n}-1)(1+|B|\theta^{2n}) > (A\theta^4-1)(1+|B|\theta^4) = |\Pi_2|.$$

Next, if n is even and negative, say $n = -m$, we have

$$\begin{aligned}|\Pi_n| &= (1-A\theta^{-2m})(1+|B|\theta^{-2m})\\ &= 1-S\theta^{-2m}-P\theta^{-4m}\\ &> 1-S-P\\ &= \Pi_0.\end{aligned}$$

Thirdly, if n is odd and negative, say $n = -m$, we have

$$|\Pi_n| = (A\theta^{-2m}+1)(1-|B|\theta^{-2m}) = 1+S\theta^{-2m}-P\theta^{-4m}.$$

To prove that this exceeds $\Pi_1 = 1+S\theta^2-P\theta^4$, it suffices to prove that

$$P(\theta^4-\theta^{-4m}) > S(\theta^2-\theta^{-2m}).$$

By (9), this is the same as

$$\theta^4-\theta^{-4m} > (\theta^2-1)(\theta^2-\theta^{-2m}),$$

which reduces to

$$(\theta^2-\theta^{-2m})(1+\theta^{-2m}) > 0,$$

and is obviously satisfied for $m \geqslant 1$.

There remains now only the case in which n is odd and greater than 1. Since $\theta^6 = 17{\cdot}9...$, it follows that $|B|\theta^{2n} > 1$. Hence

$$\begin{aligned}|\Pi_n| &= (A\theta^{2n}+1)(|B|\theta^{2n}-1)\\ &\geqslant (A\theta^6+1)(|B|\theta^6-1)\\ &> (17A+1)(17|B|-1),\end{aligned}$$

and this, by (11), is much larger than $5/(1+2\sqrt{5})$. This completes the proof of Lemma 2.

It may be remarked that this lemma does not depend in any essential way on the precise value of θ. It remains true for any number of about the same magnitude as θ, except that the numbers in (4) and (5) must be given their general form as functions of θ.

3. Proof of the first assertion of the Theorem

Let x and y be any linear forms in u and v of determinant 1. If the linear form x represents zero for integral values of u and v, not both zero, there is nothing to prove. Hence we can write

$$x = \lambda(u-\phi v),$$

where $\lambda \neq 0$ and ϕ is irrational. We can also write

$$\lambda y = v+\mu(u-\phi v),$$

since the determinant of the linear forms x, y is 1.

We appeal to Lemma 1. If the first alternative of that lemma arises, there are infinitely many approximations u/v to ϕ which satisfy (2). For these approximations,

$$x^2y^2-x^4 = (u-\phi v)^2\{v+\mu(u-\phi v)\}^2-\lambda^4(u-\phi v)^4$$
$$= (u-\phi v)^2v^2+o(1)$$

as $u/v \to \phi$. Hence, in this case, any inequality

$$|x^2(x^2-y^2)| < K$$

with $K > \frac{1}{8}$ has an infinity of solutions in integers u, v not both zero.

We may now suppose that the second alternative in Lemma 1 arises. The substitution

$$\alpha u+\beta v = U, \qquad \gamma u+\delta v = V,$$

which has integral coefficients and determinant ± 1, transforms x into a multiple of $U-\theta V$, say

$$x = \rho(U-\theta V).$$

Let $\theta' = \frac{1}{2}(1-\sqrt{5})$ be the algebraic conjugate of θ. We can write

$$y = \sigma(U-\theta V)+\tau(U-\theta' V)$$

for some σ, τ. By comparison of determinants, we have

$$\pm 1 = \rho\tau(\theta-\theta'), \quad \text{i.e.} \quad |\rho\tau| = \frac{1}{\sqrt{5}}.$$

On factorizing x^2-y^2, we get

$$x^2(x^2-y^2)$$
$$= -\tfrac{1}{5}(U-\theta V)^2\{A(U-\theta V)-(U-\theta' V)\}\{B(U-\theta V)-(U-\theta' V)\}.$$

Here
$$A = -\frac{\rho+\sigma}{\tau}, \qquad B = \frac{\rho-\sigma}{\tau},$$

and so A and B are not both zero, since ρ is not zero.

Every integral power of θ is representable as

$$\theta^n = U-\theta V,$$

where U, V are integers. Since $\theta' = -\theta^{-1}$, we have

$$(-\theta)^{-n} = U-\theta' V.$$

Giving U, V these values, we obtain

$$|x^2(x^2-y^2)| = \tfrac{1}{5}|\{A(-\theta^2)^n-1\}\{B(-\theta^2)^n-1\}|.$$

By Lemma 2 we can choose the integer n so that the right-hand side does not exceed

$$\frac{1}{1+2\sqrt{5}},$$

and this establishes the first assertion of the theorem.

4. For the proof of the second assertion of the theorem, we need a further lemma.

LEMMA 3. *If u and v are integers, the number*

$$u^2+4\theta^{-2}uv+\theta^{-2}v^2 \tag{13}$$

is a unit of the quadratic field $k(\sqrt{5})$ if and only if (u, v) is one of the pairs

$$\pm(0,1),\quad \pm(1,0),\quad \pm(1,-1),\quad \pm(1,-3),\quad \pm(1,-4). \tag{14}$$

Proof. The number (13) is a unit of $k(\sqrt{5})$ if and only if

$$u^4+3u^2(4uv+v^2)+(4uv+v^2)^2 = \pm 1. \tag{15}$$

If we write $x = 2u+v$, $y = u$, this becomes

$$x^4-5x^2y^2+5y^4 = \pm 1. \tag{16}$$

On considering the parities of x and y, we see that the left-hand side is always congruent to 0 or 1 (mod 4). Hence the equation with -1 on the right is insoluble. The equation with 1 can be written

$$5y^4 = (2x^2-5y^2+2)(2x^2-5y^2-2). \tag{17}$$

The possibility $y = 0$ leads to $x = \pm 1$, and so to $u = 0$, $v = \pm 1$. We can suppose henceforward that $y > 0$, $x > 0$.

The highest common factor d of the two numbers on the right of (17) can only be 1 or 2 or 4. It is impossible that d should be 2. For this would require y to be even, and on writing $y = 2Y$, we should have

$$20Y^4 = (x^2-10Y^2+1)(x^2-10Y^2-1).$$

But now the two factors on the right are not relatively prime, since they have the same parity and their product is even.

Case 1. Suppose $d = 1$. The general expression of $5y^4$ as the product of two relatively prime factors is

$$5y^4 = (\pm 5p^4)(\pm q^4), \quad \text{where} \quad (5p, q) = 1.$$

Using this in (17), we obtain

$$5p^4-q^4 = \pm 4. \tag{18}$$

A result due to Ljunggren* tells us that of the two equations

$$5p^4-q^4 = 4, \qquad 5p^4-q^4 = -4,$$

at most one is soluble in positive integers, and, if one is soluble, it has only one solution. Hence the only solution of (18) in positive integers is $p = 1$, $q = 1$. This implies $y = 1$. Now (16) gives

$$x^4-5x^2 = -4,$$

* See Skolem, *Diophantische Gleichungen*, 113 (Satz 30*a*, Satz 30*b*).

whence $x = 1$ or 2. Allowing for changes of sign, we obtain the pairs

$$(u, v) = \pm(1, -1),\ \pm(1, 0),\ \pm(1, -3),\ \pm(1, -4).$$

Case 2. Suppose $d = 4$. Here y must be even and x odd. Writing $y = 2Y$, $x = 2X+1$, we obtain from (17)

$$5Y^4 = (2X^2+2X-5Y^2+1)(2X^2+2X-5Y^2).$$

Since the factors on the right are relatively prime, one of them is $\pm 5p^4$ and the other $\pm q^4$. We obtain

$$5p^4-q^4 = \pm 1.$$

The same result due to Ljunggrens tells us that the only solution is $p = 2$, $q = 3$. This gives $Y = 6$, whence $y = 12$. The equation (16), with 1 on the right, becomes

$$x^4-5(12)^2x^2+5(12)^4-1 = 0.$$

This gives $x^2 = 521$ or 199, both of which are impossible.

5. Proof of the second assertion of the Theorem

Let A, B be the particular real numbers defined by (5). Let x, y be the special linear forms

$$\left.\begin{aligned} x &= \left(\frac{A-B}{2\sqrt{5}}\right)^{\frac{1}{2}}(u-\theta v) \\ y &= \left(\frac{2}{(A-B)\sqrt{5}}\right)^{\frac{1}{2}}\{(u-\theta' v)-\tfrac{1}{2}(A+B)(u-\theta v)\} \end{aligned}\right\}. \tag{19}$$

Their determinant is 1, since $\theta-\theta' = \sqrt{5}$. We have

$$x^2(x^2-y^2)$$
$$= -\tfrac{1}{5}(u-\theta v)^2\{A(u-\theta v)-(u-\theta' v)\}\{B(u-\theta v)-(u-\theta' v)\}, \tag{20}$$

i.e. $$|x^2(x^2-y^2)| = \tfrac{1}{5}\xi^2|(A\xi-\xi')(B\xi-\xi')|, \tag{21}$$

where ξ is a general integer of $k(\sqrt{5})$ and ξ' denotes its algebraic conjugate. We have to prove that the right-hand side of (21) is always greater than or equal to $(1+2\sqrt{5})^{-1}$ when $\xi \neq 0$. We already know from Lemma 2 that this is true when ξ is a unit of $k(\sqrt{5})$; for, if $\xi = \pm\theta^n$, the right-hand side of (21) is

$$\tfrac{1}{5}|\{A(-\theta^2)^n-1\}\{B(-\theta^2)^n-1\}| = \tfrac{1}{5}|\Pi_n| \geqslant (1+2\sqrt{5})^{-1}.$$

We may therefore assume henceforward that

$$|\xi\xi'| \geqslant 4, \tag{22}$$

since 4 is the least absolute norm of an integer other than a unit.

The product of the last two factors on the right of (20) is a certain indefinite binary quadratic form in u and v. The coefficients in this form are easily calculated by using (12), which tells us the value of the form for three particular pairs of values of u and v. Denoting the binary form in question by $au^2+buv+cv^2$, we obtain, taking $u = 1$, $v = 0$,

$$a = (A-1)(B-1) = \Pi_0 = \frac{5}{1+2\sqrt{5}};$$

taking $u = 0$, $v = 1$,

$$c = (A\theta+\theta^{-1})(B\theta+\theta^{-1}) = \theta^{-2}\Pi_1 = \frac{5\theta^{-2}}{1+2\sqrt{5}};$$

and taking $u = 1$, $v = -1$,

$$a-b+c = (A\theta^2-\theta^{-2})(B\theta^2-\theta^{-2}) = \theta^{-4}\Pi_2 = -\frac{5\theta^{-4}}{1+2\sqrt{5}}.$$

It follows that

$$b = 5\left(\frac{1+\theta^{-2}+\theta^{-4}}{1+2\sqrt{5}}\right) = \frac{20\theta^{-2}}{1+2\sqrt{5}}.$$

Hence (20) can be written

$$x^2(x^2-y^2) = -\phi(u,v)/(1+2\sqrt{5}), \tag{23}$$

where $$\phi(u,v) = (u-\theta v)^2\{u^2+4\theta^{-2}uv+\theta^{-2}v^2\}.$$

We have now to prove that $|\phi(u,v)| \geqslant 1$ for all integers u and v, other than $0, 0$.

We have already seen that we may assume that $\xi = u-\theta v$ satisfies (22), which implies that

$$(u-\theta v)^2(u+\theta^{-1}v)^2 \geqslant 16. \tag{24}$$

The next step is to observe that we may also suppose that the number (13) is not a unit of $k(\sqrt{5})$. By Lemma 3 it suffices to examine the pairs of values for u and v listed in (14). Of these, the pairs $\pm(0,1)$, $\pm(1,0)$, $\pm(1,-1)$ can be rejected at once, since then $u-\theta v$ is a unit of $k(\sqrt{5})$, and so does not satisfy (24). There remain the pairs $\pm(1,-3)$ and $\pm(1,-4)$. Now

$$|\phi(1,-3)| = (1+3\theta)^2(3\theta^{-2}-1) = 5,$$

$$|\phi(1,-4)| = (1+4\theta)^2 = 55{\cdot}8\ldots.$$

Hence the result holds in these cases.

We may now assume that the number (13) is not a unit, which implies that

$$|(u^2+4\theta^{-2}uv+\theta^{-2}v^2)(u^2+4\theta^2uv+\theta^2v^2)| \geqslant 4. \tag{25}$$

The inequalities (24) and (25) allow us to write down two lower bounds for $|\phi(u,v)|$, namely

$$|\phi(u,v)| \geqslant \frac{16|u^2+4\theta^{-2}uv+\theta^{-2}v^2|}{(u+\theta^{-1}v)^2}$$

and
$$|\phi(u,v)| \geqslant \frac{4(u-\theta v)^2}{|u^2+4\theta^2uv+\theta^2v^2|}.$$

As we can obviously suppose that $v \neq 0$, we write $r = u/v$, and have to prove that one of the two inequalities

$$16|r^2+4\theta^{-2}r+\theta^{-2}| \geqslant (r+\theta^{-1})^2, \tag{26}$$

$$4(r-\theta)^2 \geqslant |r^2+4\theta^2r+\theta^2|, \tag{27}$$

holds for any real number r.

Suppose first that $r \geqslant 0$. Then

$$16r^2+64\theta^{-2}r+16\theta^{-2} > r^2+2\theta^{-1}r+\theta^{-2},$$

and consequently (26) holds. Suppose next that $r < 0$, and write $r = -s$. If $s \geqslant 2$, we have $15s^2-64\theta^{-2}s > 0$ and

$$\begin{aligned}16s^2-64\theta^{-2}s+16\theta^{-2} &> s^2+16\theta^{-2}\\ &> s^2-2\theta^{-1}s+\theta^{-2},\end{aligned}$$

so again (26) holds. If $0 < s < 2$, we have

$$\theta^2 > s^2-4\theta^2s+\theta^2 > 4-8\theta^2+\theta^2,$$

$$|s^2-4\theta^2s+\theta^2| < 7\theta^2-4 < 8 < 4(s+\theta)^2,$$

so that in this case (27) holds. This completes the proof that $|\phi(u,v)| \geqslant 1$ for all integers u, v other than $0, 0$; and so establishes the fact that there is no solution of (1) with strict inequality for the special linear forms defined in (19).

INDEFINITE TERNARY QUADRATIC FORMS

By H. BLANEY (*London*)

[Received 19 September 1949]

1. LET $Q(x,y,z)$ be an indefinite ternary quadratic form with real coefficients and non-zero determinant D. Various inequalities are known, with best-possible constants, which are satisfied by either

(i) $Q(x,y,z)$ for integral x, y, z with $(x,y,z) = 1$, or

(ii) $Q(x+x_0, y+y_0, z+z_0)$ for integral x, y, z, where x_0, y_0, z_0 are any real numbers.

Of the type (i), there is Markoff's theorem (**1**)*

$$|Q(x,y,z)| \leqslant (\tfrac{2}{3}|D|)^{\frac{1}{3}},$$

and there are the inequalities due to Davenport (**3**)

$$0 < Q(x,y,z) \leqslant (-4D)^{\frac{1}{3}} \quad (D < 0), \tag{1}$$

$$0 < Q(x,y,z) \leqslant (\tfrac{27}{4}D)^{\frac{1}{3}} \quad (D > 0). \tag{2}$$

Of the type (ii), there is Davenport's theorem (**4**)

$$|Q(x+x_0, y+y_0, z+z_0)| \leqslant (\tfrac{27}{100}|D|)^{\frac{1}{3}}.$$

There remains the question of the non-homogeneous analogues to (1) and (2). My main object in this note is to give the first of these analogues by proving the following

THEOREM. *Let $Q(x,y,z)$ be an indefinite ternary quadratic form with determinant $D < 0$. Then, for any real x_0, y_0, z_0, there exist integers x, y, z such that*

$$0 < Q(x+x_0, y+y_0, z+z_0) \leqslant (-4D)^{\frac{1}{3}}.$$

This is true with strict inequality unless $Q(x+x_0, y+y_0, z+z_0)$ is equivalent† *to a positive multiple of either*

$$x^2+yz \quad \text{or} \quad (x-y)(x+y+1)+2z^2,$$

in which cases it is not.

* For an elementary proof, see Davenport (**2**).

† We say that $Q(x+x_0, y+y_0, z+z_0)$ is equivalent ($\sim$) to another form of this kind if one can be derived from the other by a certain integral unimodular substitution on both x, y, z and x_0, y_0, z_0 simultaneously, together with the addition of arbitrary integers to x_0, y_0, z_0. Thus a non-homogeneous form equivalent to x^2+yz is necessarily one for which x_0, y_0, z_0 are integers.

Quart. J. Math. Oxford (2), 1 (1950), 262–9.

The proof is based on the familiar principle of selecting a special value of $Q(x,y,z)$, with $(x,y,z) = 1$, and transforming so that this becomes $Q(1,0,0)$. The selection is made by using a modified form of (1), which is given in Lemma 1. The subsequent argument requires the use of certain asymmetric inequalities satisfied by a non-homogeneous indefinite binary quadratic form (Lemmas 2 and 3).

When $D > 0$, we have the problem of finding the exact non-homogeneous analogue to (2), and this seems to be a more difficult question. It is possible to prove a result with $3D^{\frac{1}{3}}$ on the right, and the following example shows that the number on the right must be at least $2D^{\frac{1}{3}}$, which, incidentally, is greater than $(\frac{27}{4}D)^{\frac{1}{3}}$. Consider the non-homogeneous form

$$(x-y)(x+y+z+1)-z^2.$$

This has $D = 1$, and it is readily seen, by considering the parities of z and of $x-y$, that this form does not represent unity. Hence this form cannot satisfy an inequality analogous to (2) with a number less than $2D^{\frac{1}{3}}$ on the right.

I am grateful to Professor Davenport for his advice concerning the presentation of this note.

2. I first prove that the number zero on the left of (1) can be replaced by a certain positive number without disturbing the number on the right. This is not an essential step for the proof of the theorem, but it effects a slight simplification.

LEMMA 1. *Let $Q(x,y,z)$ be a real indefinite ternary quadratic form with determinant $D < 0$. Then there exist a positive number δ and integers x, y, z, with $(x,y,z) = 1$, such that*

$$\delta(-4D)^{\frac{1}{3}} < Q(x,y,z) \leqslant (-4D)^{\frac{1}{3}}. \tag{3}$$

This is certainly true for $\delta = 0{\cdot}04$; and with strict inequality on the right unless (with $\lambda > 0$)

$$Q(x,y,z) \sim \lambda[x^2+yz],$$

in which case equality is needed.

Proof. I shall deduce the result from Theorem 1 of (**5**), in the case $n = 3$, and from the final equation in the proof of that theorem [(**5**) § 4].

Let p_3 be any constant such that the inequality

$$0 < Q(x,y,z) \leqslant p_3|D|^{\frac{1}{3}}$$

is always soluble in integers x, y, z, and let p_2 be a similar constant for indefinite binary forms. It was proved (loc. cit.) that, for any $c \geqslant 0$, it is possible to satisfy the inequality

$$c|D|^{\frac{1}{3}} < Q(x,y,z) \leqslant C|D|^{\frac{1}{3}}$$

in integers x, y, z, with $(x,y,z) = 1$, where

$$C = \max[p_3, 2c+2(c^2+p_2 c^{\frac{1}{2}})^{\frac{1}{2}}].$$

The value $p_3 = 4^{\frac{1}{3}}$ is admissible, by (1), and it is classical that $p_2 = 2$ is admissible. [See, e.g., (**5**) Lemma 1.] Hence, if we choose c so that

$$4^{\frac{1}{3}} = 2c+2(c^2+2c^{\frac{1}{2}})^{\frac{1}{2}}, \tag{4}$$

the desired result (3) will follow with $\delta = 4^{\frac{1}{3}}c$. We find, from (4), that

$$c = \tfrac{1}{4}(3-2\sqrt{2})4^{\frac{1}{3}},$$

which shows that (3) holds with $\delta = \frac{1}{4}(3-2\sqrt{2}) > 0{\cdot}04$.

The rest of the lemma is really due to Davenport (**3**) who proved that equality is required in (1) if and only if $Q(x,y,z) \sim \lambda(x^2+yz)$ $(\lambda > 0)$. By the proof of Theorem 1 of (**5**) we need only take $\delta < \frac{1}{4}(3-2\sqrt{2})$ to ensure that the same result still holds for the inequality (3). This proves the lemma.

3. We now require two lemmas concerning non-homogeneous indefinite binary quadratic forms. Let $\phi(y,z)$ be a real indefinite binary quadratic form with discriminant d (> 0), and let y_0, z_0 be any real numbers.

LEMMA 2. *For any $k > 0$, there exist integers y, z such that*

$$-\left(\frac{d}{4k}\right)^{\frac{1}{2}} < \phi(y+y_0, z+z_0) \leqslant \left(\frac{kd}{64}\right)^{\frac{1}{2}}.$$

This result is due to Davenport [(**4**) Lemma 4]. The product of the two numbers is $d/16$.

LEMMA 3. *There exist integers y, z such that*

$$-(d/128)^{\frac{1}{2}} < \phi(y+y_0, z+z_0) \leqslant 7(d/128)^{\frac{1}{2}}.$$

This is true with strict inequality unless (with $\lambda > 0$)

$$\phi(y+y_0, z+z_0) \sim \lambda[2z^2-(y+\tfrac{1}{2})^2],$$

in which case it is not.

Proof. This is the special case $\mu = \frac{1}{7}$ $(n = 2)$, of Theorem 2 of (**6**). (It is also the case $q^2 = 32$ $(m = 1)$, of Theorem 4 of the same.) The result is more precise than that of Lemma 2 for the special interval in question, since the product of the two numbers is $7d/128$ $(< d/16)$.

4. In order to prove the theorem we may, without loss of generality, suppose that $D = -\frac{1}{4}$. By Lemma 1, there is a value a of $Q(x,y,z)$, with $(x,y,z) = 1$, which satisfies

$$\delta < a \leqslant 1. \tag{5}$$

After applying an integral unimodular transformation, we can suppose that $a = Q(1,0,0)$ and write

$$Q(x,y,z) = a(x+hy+gz)^2+\phi(y,z).$$

Here, $\phi(y,z)$ is an indefinite binary quadratic form, and, by comparison of determinants, the discriminant of ϕ has the value $d = 1/a$. Further, by making the substitution

$$x = \pm x'+my'+nz', \qquad y = y', \qquad z = z',$$

where m, n are suitable integers, we can ensure that

$$0 \leqslant h \leqslant \tfrac{1}{2}, \qquad |g| \leqslant \tfrac{1}{2}. \tag{6}$$

We now have

$$Q = Q(x+x_0, y+y_0, z+z_0) = a(x+\alpha)^2+\phi(y+y_0, z+z_0), \tag{7}$$

subject to (5) and (6), where

$$\alpha = x_0+h(y+y_0)+g(z+z_0) \tag{8}$$

is independent of x. Further, by our notion of equivalence, we may suppose, without loss of generality, that, in (7),

$$0 \leqslant x_0, y_0, z_0 < 1. \tag{9}$$

To prove the theorem we have to show that integers x, y, z exist such that

$$0 < Q = Q(x+x_0, y+y_0, z+z_0) \leqslant 1, \tag{10}$$

and to determine the cases in which the sign of equality is actually needed.

5. Before considering the general case, I first prove two lemmas relating to the special cases in which equality is needed in (10).

LEMMA 4. *If $a = \frac{1}{2}$ and*

$$\phi(y+y_0, z+z_0) \sim z^2-\tfrac{1}{2}(y+\tfrac{1}{2})^2,$$

then $Q(x+x_0, y+y_0, z+z_0)$ satisfies (10) *with strict inequality unless*

$$Q(x+x_0, y+y_0, z+z_0) \sim \tfrac{1}{2}(x+\tfrac{1}{2})^2-\tfrac{1}{2}(y+\tfrac{1}{2})^2+z^2,$$

in which case it does not.

Proof. By making the appropriate substitution, we can in fact suppose that

$$\phi(y+y_0, z+z_0) = z^2-\tfrac{1}{2}(y+\tfrac{1}{2})^2.$$

By Lemma 3 with $d = 2$, $\phi(y+y_0, z+z_0)$ assumes the values $-\frac{1}{8}$ and $\frac{7}{8}$, but no intermediate value.

More explicitly, $\phi = -\frac{1}{8}$ for $z = 0$, $y = 0$ or -1. For either pair of values of y, z, choose x so that $\frac{1}{2} \leqslant |x+\alpha| \leqslant 1$. We then have

$$0 \leqslant Q = \tfrac{1}{2}(x+\alpha)^2-\tfrac{1}{8} \leqslant \tfrac{3}{8} < 1,$$

which is (10), and with strict inequality, unless $Q = 0$. This can only be the case if $|x+\alpha| = \frac{1}{2}$. Now x is an integer, hence, by (8), taking account of both values of y, we must have

$$\alpha = x_0 \pm \tfrac{1}{2}h \equiv \tfrac{1}{2} \pmod 1$$

in each case. Hence, by (6) and (9), these two equations give us

$$h = 0, \qquad x_0 = \tfrac{1}{2}. \tag{11}$$

Again, $\phi = \frac{7}{8}$ for $y = 0$, $z = 1$, and, choosing x so that $|x+\alpha| \leqslant \frac{1}{2}$, we now have

$$0 < \tfrac{7}{8} \leqslant Q = \tfrac{1}{2}(x+\alpha)^2 + \tfrac{7}{8} \leqslant 1,$$

which is (10) with strict inequality unless $Q = 1$, so that $|x+\alpha| = \frac{1}{2}$. This requires, by (8) and (11),

$$\alpha = \tfrac{1}{2} + g \equiv \tfrac{1}{2} \pmod 1,$$

whence, by (6), $g = 0$.

We have now shown that, in the present case, (10) has a solution with strict inequality except perhaps when

$$\begin{aligned} Q(x+x_0, y+y_0, z+z_0) &= \tfrac{1}{2}(x+\tfrac{1}{2})^2 - \tfrac{1}{2}(y+\tfrac{1}{2})^2 + z^2 \\ &= \tfrac{1}{2}(x-y)(x+y+1) + z^2. \end{aligned}$$

Clearly $(x-y)(x+y+1)$ is always even, so that this ternary form assumes only integral values and does require equality in (10).

This completes the proof of the lemma.

Lemma 5. *If $a = 1$, then necessarily*

$$Q(x+x_0, y+y_0, z+z_0) \sim (x+x_0)^2 + (y+y_0)(z+z_0),$$

and such a form satisfies (10) *with strict inequality unless, in fact,*

$$Q(x+x_0, y+y_0, z+z_0) \sim x^2 + yz,$$

in which case it does not.

Proof. By Lemma 1, if $a = 1$ in (5), that is, if (3) has no solution with strict inequality, $Q(x, y, z) \sim x^2 + yz$. Hence, by making the appropriate substitution, we can, in fact, suppose that

$$Q = Q(x+x_0, y+y_0, z+z_0) = (x+x_0)^2 + (y+y_0)(z+z_0).$$

Choose y so that $|y+y_0| \leqslant \frac{1}{2}$, and then choose z so that

$$0 < |z+z_0| \leqslant 1 \quad \text{and} \quad (y+y_0)(z+z_0) \geqslant 0.$$

We then have

$$0 \leqslant (y+y_0)(z+z_0) \leqslant \tfrac{1}{2} \tag{12}$$

with strict inequality on the left unless, by (9), $y_0 = 0$. If this is the case, choose z so that $|z+z_0| \leqslant \frac{1}{2}$, and then take $y = \pm 1$ so that $y(z+z_0) \geqslant 0$. We then have a solution of (12) with strict inequality on

the left unless $z_0 = 0$, also. Hence, finally, we have (12) with strict inequality on the left unless $y_0 = z_0 = 0$.

With this final choice of y, z we now choose x so that $|x+x_0| \leqslant \frac{1}{2}$. We then have

$$0 \leqslant Q = (x+x_0)^2+(y+y_0)(z+z_0) \leqslant \tfrac{1}{4}+\tfrac{1}{2} = \tfrac{3}{4} < 1,$$

which is (10), and with strict inequality, unless $Q = 0$, so that

$$|x+x_0| = (y+y_0)(z+z_0) = 0.$$

By (9) and the previous argument, these conditions require

$$x_0 = y_0 = z_0 = 0.$$

Hence $$Q(x+x_0, y+y_0, z+z_0) \sim x^2+yz.$$

Finally, by Davenport (3) (loc. cit.), the form x^2+yz does require equality in (1) and thus also in (10). In fact, this form clearly assumes only integral values.

This completes the proof of the lemma.

6. Proof of the theorem

It suffices to show that (10) is satisfied with complete inequality except when the conditions of Lemmas 4 and 5 obtain.

Case (i): $\delta < a < \frac{1}{2}$. We can choose a positive integer r so that

$$2^{r-1}+1 < 1/a \leqslant 2^r+1. \tag{13}$$

Moreover, $$2^{r-1}+1 < 1/a < 1/\delta < 25,$$

so $r \leqslant 5$, but this is not essential to the proof.

By Lemma 2 with $d = 1/a$ and

$$k = 4(2^{r-1}+1)^{-4}a^{-3},$$

there exist integers y, z such that

$$-\tfrac{1}{4}(2^{r-1}+1)^2 a < \phi(y+y_0, z+z_0) \leqslant \tfrac{1}{4}(2^{r-1}+1)^{-2}a^{-2}. \tag{14}$$

This choice of y, z fixes the value of α, by (8).

Suppose first that $\phi(y+y_0, z+z_0) > 0$ in (14), and choose x so that $|x+\alpha| \leqslant \frac{1}{2}$. We then have $Q > 0$ and, by (14),

$$Q \leqslant \tfrac{1}{4}a+\tfrac{1}{4}(2^{r-1}+1)^{-2}a^{-2}.$$

This function of a has a negative derivative since, by (13),

$$a^3 < (2^{r-1}+1)^{-3} < 2(2^{r-1}+1)^{-2}.$$

Hence
$$Q \leqslant \frac{1}{4(2^r+1)}+\frac{(2^r+1)^2}{4(2^{r-1}+1)^2}$$

$$= \frac{4.2^{3r}+13.2^{2r}+16.2^r+8}{4.2^{3r}+20.2^{2r}+32.2^r+16} < 1.$$

Suppose now that $\phi(y+y_0, z+z_0) \leqslant 0$ in (14). Then there exists an integer s such that

$$-\tfrac{1}{4}s^2 a < \phi(y+y_0, z+z_0) \leqslant -\tfrac{1}{4}(s-1)^2 a, \tag{15}$$

where, by (14),
$$1 \leqslant s \leqslant 2^{r-1}+1. \tag{16}$$

Now choose x so that
$$\tfrac{1}{2}s \leqslant |x+\alpha| \leqslant \tfrac{1}{2}(s+1).$$

We then have, by (13), (15), (16),

$$\begin{aligned} 0 < Q &\leqslant \tfrac{1}{4}(s+1)^2 a - \tfrac{1}{4}(s-1)^2 a = sa \\ &\leqslant (2^{r-1}+1)a \\ &< 1, \end{aligned}$$

which is (10) with strict inequality.

Case (ii): $\tfrac{1}{2} \leqslant a \leqslant 1$. By Lemma 3 with $d = 1/a$, there exist integers y, z such that

$$-1/\{8(2a)^{\frac{1}{2}}\} < \phi(y+y_0, z+z_0) \leqslant 7/\{8(2a)^{\frac{1}{2}}\}. \tag{17}$$

The sign of equality is needed if and only if

$$\phi(y+y_0, z+z_0) \sim (8a)^{-\frac{1}{2}}\{2z^2 - (y+\tfrac{1}{2})^2\}. \tag{18}$$

Suppose first that $\phi(y+y_0, z+z_0) > 0$ in (17), and choose x so that $|x+\alpha| \leqslant \tfrac{1}{2}$. We then have $Q > 0$ and, by (17),

$$Q \leqslant \tfrac{1}{4}a + 7/\{8(2a)^{\frac{1}{2}}\} = g(a),$$

say. Now
$$g'(a) = \tfrac{1}{4} - 7/\{8(2a)^{\frac{3}{2}}\} < 0,$$

since $a^3 \leqslant 1 < 49/32$. Moreover, $g(\tfrac{1}{2}) = 1$. Hence $0 < Q \leqslant 1$, and equality occurs only if $a = \tfrac{1}{2}$, $|x+\alpha| = \tfrac{1}{2}$ and

$$\phi(y+y_0, z+z_0) = 7/\{8(2a)^{\frac{1}{2}}\}.$$

This last condition requires $\phi(y+y_0, z+z_0)$ to be equivalent to the special form (18). By Lemma 4, it follows that, for equality in (10),

$$Q(x+x_0, y+y_0, z+z_0) \sim \tfrac{1}{2}(x-y)(x+y+1)+z^2.$$

Suppose now that $\phi(y+y_0, z+z_0) \leqslant 0$ in (17), and choose x so that $\tfrac{1}{2} \leqslant |x+\alpha| \leqslant 1$. We then have, since $a \geqslant \tfrac{1}{2}$,

$$Q > \tfrac{1}{4}a - 1/\{8(2a)^{\frac{1}{2}}\} \geqslant 0,$$

and $Q \leqslant a \leqslant 1$. Hence $0 < Q \leqslant 1$, and equality occurs only if $a = 1$. By Lemma 5, it follows that, for equality in (10),

$$Q(x+x_0, y+y_0, z+z_0) \sim x^2 + yz.$$

This completes the proof of the theorem.

REFERENCES

1. H. Markoff, *Math. Annalen*, 56 (1903), 233–51.
2. H. Davenport, 'On a theorem of Markoff', *J. of London Math. Soc.* 22 (1947), 96–9.
3. —— 'On indefinite ternary quadratic forms', *Proc. London Math. Soc.* 51 (1949), 145–60.
4. —— 'Non-homogeneous ternary quadratic forms', *Acta Math.* 80 (1948), 65–95.
5. H. Blaney, 'Indefinite quadratic forms in *n* variables', *J. of London Math. Soc.* 23 (1948), 153–60.
6. —— 'Some asymmetric inequalities', *Proc. Cambridge Phil. Soc.* 46 (1950), 359–76.

THE REPRESENTATION OF A FINITE GROUP AS A GROUP OF AUTOMORPHISMS ON A FINITE ABELIAN GROUP

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[Received 21 September 1949]

1. Introduction

THE subject of this paper is the representation of abstract finite groups by automorphisms of finite abelian groups. The latter groups will be written in an additive notation, i.e. as modules. The problem can thus be reduced to that of representations by matrices whose elements lie in a residue-ring *modulo* a prime power in the domain of rational integers. In the present paper I limit myself to the cases where the orders of the abstract group and of the module are coprime. A close parallelism to representations in Galois fields GF(p) will be seen to exist.

In a future paper, I hope to apply the results of this paper to a study of the class-groups of self-conjugate algebraic number fields.

I am grateful to Professor H. Heilbronn for encouragement and valuable discussions.

2. Notation

I use small italics, except those mentioned below, for integers $\geqslant 0$; p for prime numbers. I write $\mathfrak{r}(p^r)$ for the residue ring (mod p^r) in the domain of rational integers, a, b for its elements; 0, 1 for its zero and unit-element respectively; p for the element in $\mathfrak{r}(p^r)$ which corresponds to the prime number p. $\mathfrak{M}$, $\mathfrak{N}$ denote modules; u, v their elements, $\bar{0}$ their zero element, and $\bar{u}$ a base of a module.

Greek capitals are used for sets of endomorphisms and for abstract groups and group rings; small Greek letters for their elements and in general for mappings and endomorphisms: in particular ι for the identical mapping. Gothic capitals, except those mentioned above, denote sets, rings, and groups of matrices. Capital italics are used for matrices; in particular Z for zero matrices, E for unit matrices, I for the unique unit matrix of a given set of matrices.

Furthermore: (a_{ki}) denotes the matrix with the general element a_{ki}. (*) the ideal generated by *.

The union of two algebraic systems X and Y is denoted by $X+Y$, their direct sum by $X\dot{+}Y$. The sign $\sim$ denotes a homomorphism, the

sign $\approx$ an isomorphism, and the sign $\simeq$ an equivalence, in a sense which will be defined later.

3. The theory of abelian groups and their endomorphisms is assumed to be known [cf. (**1**), (**2**)]. This section consists of restatements and obvious extensions of known results in a form in which they will be needed later.

All modules considered in this paper are of type $(p^{s_1}, p^{s_2},..., p^{s_m})$. A module $\mathfrak{M}$ of this type has a ring $\mathfrak{r}(p^r)$ as a domain of multipliers, such that $1 . u = u$. Every element $u \in \mathfrak{M}$ is a linear combination of base elements $(u_1,..., u_m)$:

$$u = \sum_{i=1}^{m} a_i u_i,$$

where $a_i \in \mathfrak{r}(p^r)$, and a_i is uniquely determined $(\text{mod}\, p^{s_i})$ $(i = 1,..., m)$. The number m of base elements (the *dimension*) and the set of annihilating ideals $\{(p^{s_1}),..., (p^{s_m})\}$, apart from their ordering, are invariants of $\mathfrak{M}$.

Notation. Sets of m ideals in $\mathfrak{r}(p^r)$ will be denoted by small gothic letters, in particular the set $\{(p^l),..., (p^l)\}$ by $\mathfrak{p}^l$.

Referred to a given base $\bar{u} = (u_1,..., u_m)$ of a module $\mathfrak{M}$, the set of ideals $\mathfrak{a} = \{(p^{l_1}),..., (p^{l_m})\}$ determines a sub-module $\mathfrak{N} = \mathfrak{a}\mathfrak{M}$ generated by $(p^{l_1}u_1,..., p^{l_m}u_m)$. Conversely, if $\mathfrak{N}$ is a sub-module of $\mathfrak{M}$, $\exists$ a base $\bar{u}$ of $\mathfrak{M}$ and a unique set of ideals $\mathfrak{a}$ such that $\mathfrak{N} = \mathfrak{a}\mathfrak{M}$.

If Δ is any set of endomorphisms on a module $\mathfrak{M}$, and $\bar{u}$ a base of $\mathfrak{M}$, then the system of equations

$$\delta u_i = \sum_{k=1}^{m} a_{ki}(\delta) u_k, \qquad \delta \in \Delta \quad (i = 1,..., m) \tag{3.1}$$

defines a *faithful* representation of Δ

$$\delta \to T(\delta) = (a_{ki}(\delta)) \tag{3.2}$$

by square matrices with elements in $\mathfrak{r}(p^r)$.

For δ, $\gamma \in \Delta$, $a \in \mathfrak{r}(p^r)$

$$T(\gamma+\delta) = T(\gamma)+T(\delta); \quad T(\gamma\delta) = T(\gamma)T(\delta); \quad T(a\delta) = T(\delta a) = aT(\delta). \tag{3.3}$$

Matrices representing endomorphisms (automorphisms) will be called e-matrices (a-matrices). K. Shoda first stated and proved the following theorems on e-matrices [cf. (**3**) § 1].

THEOREM 3.1. *If $\mathfrak{M}$ is a module of type $(p^{s_1},..., p^{s_m})$ and if $s_i \leqslant s_{i+1}$,† $s_m = r$, then the matrix T with elements in $\mathfrak{r}(p^r)$ is an e-matrix for $\mathfrak{M}$ if and*

† This ordering of the set of annihilating ideals appears only in this theorem in order to permit a clear formulation.

only if it is of the form

$$T = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdot \\ p^{s_2-s_1}b_{21} & b_{22} & b_{23} & \cdot \\ p^{s_3-s_1}b_{31} & p^{s_3-s_2}b_{32} & b_{33} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \tag{3.4}$$

If $T = (a_{ki})$, $T' = (a'_{ki})$ are two such e-matrices for $\mathfrak{M}$ then they represent the same endomorphism on $\mathfrak{M}$, referred to a fixed base of $\mathfrak{M}$, if and only if

$$a_{ki} \equiv a'_{ki} \ (p^{s_i}) \quad (i, k = 1,..., m). \tag{3.5}$$

THEOREM 3.2. *If $\mathfrak{M}$ is a module of type $(p^{s_1},..., p^{s_m})$, then an e-matrix T for $\mathfrak{M}$ is an a-matrix if and only if the determinant*

$$|T| \not\equiv 0 \ (p). \tag{3.6}$$

DEFINITION 3.1. If $\mathfrak{M}$ is a module of type $(p^{s_1},..., p^{s_m})$, where the annihilating ideals are now assumed to be ordered in some way, and if T is an e-matrix referred to a base $\bar{u} = (u_1,..., u_m)$ which corresponds to the given ordering of the annihilating ideals, then T is said to be of *type* $(p^{s_1},..., p^{s_m})$.

While the 'type of module' is a non-ordered set, the 'type of an e-matrix' is an ordered set.

DEFINITION 3.2. Two sets $\mathfrak{S}$ and $\mathfrak{S}'$ of e-matrices for a module $\mathfrak{M}$ with elements in $\mathfrak{r}(p^r)$ are said to be *equivalent*, if

(i) there exists a (1, 1) correspondence θ between the whole set $\mathfrak{S}$ and the whole set $\mathfrak{S}'$;

(ii) there exist two bases $\bar{u}$ and $\bar{u}'$ of $\mathfrak{M}$, such that any matrix $S \in \mathfrak{S}$, referred to $\bar{u}$, represents the same endomorphism δ as its image matrix $S^\theta \in \mathfrak{S}'$ referred to $\bar{u}'$.

If condition (i) of this definition is satisfied, and if there exists a constant a-matrix R of the same type as $\mathfrak{S}$ such that

$$R^{-1}SR = S^\theta \quad (\text{all } S \in \mathfrak{S}),$$

then obviously $\mathfrak{S} \simeq \mathfrak{S}'$.

The converse of this is only true, however, if $\mathfrak{M}$ is of type $(p^s,..., p^s)$.

From now on any set of ideals $\mathfrak{a} = \{(p^{t_1}),..., (p^{t_m})\}$ will be assumed to satisfy the condition $t_i > 0$. Wherever in the following definitions $\mathfrak{M}$ is written as a direct sum, $\mathfrak{M} = \mathfrak{M}_1 \dot{+} \mathfrak{M}_2$, neither $\mathfrak{M}_1$ nor $\mathfrak{M}_2$ shall be the zero module. All the definitions refer to a fixed set Δ of endomorphisms.

DEFINITION 3.3. A submodule $\mathfrak{N}$ of $\mathfrak{M}$ is said to be *invariant* if $u \in \mathfrak{N}$ and $\delta \in \Delta$ imply $\delta u \in \mathfrak{N}$.

DEFINITION 3.4. A module $\mathfrak{M}$ is said to be *reducible* if it contains a proper invariant submodule; otherwise $\mathfrak{M}$ is said to be *irreducible*.

DEFINITION 3.5. A module $\mathfrak{M}$ is said to be $\mathfrak{a}$-*decomposable*, if

(i) $\mathfrak{M} = \mathfrak{M}_1 \dot{+} \mathfrak{M}_2$;

(ii) there exist bases $\bar{u}_1$ of $\mathfrak{M}_1$ and $\bar{u}_2$ of $\mathfrak{M}_2$ and a set of ideals $\mathfrak{a}$ such that, referred to the base $\bar{u}_1 \dot{+} \bar{u}_2$ of $\mathfrak{M}$, $\mathfrak{N} = \mathfrak{a}\mathfrak{M}$;

(iii) $u \in \mathfrak{M}_1$, $\delta \in \Delta$ imply $\delta u \in \mathfrak{M}_1 + \mathfrak{N}$.

DEFINITION 3.6. A module $\mathfrak{M}$ is said to be *completely* $\mathfrak{a}$-*decomposable* if it is $\mathfrak{a}$-decomposable, and if, whenever $\mathfrak{M}_1$, $\mathfrak{M}_2$, and $\mathfrak{N}$ satisfy the conditions of Definition (3.5), we have also

(iv) $v \in \mathfrak{M}_2$, $\delta \in \Delta$ imply $\delta v \in \mathfrak{M}_2 + \mathfrak{N}$.

DEFINITION 3.7. A module $\mathfrak{M}$ is said to be *decomposable* if

(i) $\mathfrak{M} = \mathfrak{M}_1 \dot{+} \mathfrak{M}_2$; (ii) $\mathfrak{M}_1$ is invariant.†

DEFINITION 3.8. A module $\mathfrak{M}$ is said to be *completely decomposable* if it is decomposable, and if, whenever $\mathfrak{M}_1$ and $\mathfrak{M}_2$ satisfy the conditions of Definition (3.7), we have also

(iii) $\mathfrak{M}_2$ is invariant.

Definitions (3.5)–(3.8) can be expressed in terms of matrix representations. For this purpose I introduce the following notation.

If $\mathfrak{a} = \{(p^{l_1}),..., (p^{l_m})\}$, then the equivalence relation between matrices

$$(a_{ki}) \equiv (a'_{ki}) \quad (\mathfrak{a})$$

is defined by $$a_{ki} \equiv a'_{ki} \ (p^{l_i}) \ (i, k = 1,..., m).$$

In the following lemma I refer to a fixed set $\mathfrak{S}$ of e-matrices in $\mathfrak{r}(p^r)$.

LEMMA 3.1. *A module $\mathfrak{M}$ of type $(p^{s_1},..., p^{s_m})$ is* (1) $\mathfrak{a}$-*decomposable*, (2) *completely* $\mathfrak{a}$-*decomposable*, (3) *decomposable*, (4) *completely decomposable if and only if there exists a set $\mathfrak{S}'$ of e-matrices for $\mathfrak{M}$, $\mathfrak{S}' \simeq \mathfrak{S}$ such that every $T' \in \mathfrak{S}'$ is of the respective forms*

(1) $$T' \equiv \begin{pmatrix} T_{11} & T_{12} \\ Z_{21} & T_{22} \end{pmatrix} \quad (\mathfrak{a}), \tag{3.7}$$

(2) $$T' \equiv \begin{pmatrix} T_{11} & Z_{12} \\ Z_{21} & T_{22} \end{pmatrix} \quad (\mathfrak{a}), \tag{3.8}$$

(3) $$T' = \begin{pmatrix} T_{11} & T_{12} \\ Z_{21} & T_{22} \end{pmatrix}, \tag{3.9}$$

(4) $$T' = \begin{pmatrix} T_{11} & Z_{12} \\ Z_{21} & T_{22} \end{pmatrix}, \tag{3.10}$$

† If $\mathfrak{M}$ is decomposable, it is obviously $\mathfrak{a}$-decomposable for all $\mathfrak{a}$.

where T_{ik}, Z_{ik} *are sub-matrices of* T', Z_{ik} *zero matrices and* T_{ii} *square matrices.*

If these conditions are satisfied we shall say that (1) $\mathfrak{S}$ is $\mathfrak{a}$-*decomposable*, (2) $\mathfrak{S}$ is *completely* $\mathfrak{a}$-*decomposable*, (3) $\mathfrak{S}$ is *decomposable*, (4) $\mathfrak{S}$ is *completely decomposable*.

Note. Every set of endomorphisms on a module of composite order is a direct sum of sets of endomorphisms on uniquely determined maximal sub-modules of prime power order. Thus the restriction to modules of the latter kind is not a fundamental one.

4. Let Γ be an abstract group of finite order. A group $\mathfrak{G}$ of e-matrices for a module $\mathfrak{M}$ of type $(p^{s_1},\ldots,p^{s_m})$ such that $\Gamma \sim \mathfrak{G}$ will be called 'a representation $\mathfrak{G}$ of Γ in $\mathfrak{r}(p^r)$ of degree m', where r is defined by $r = \max s_i$. $\mathfrak{M}$ will be called a 'representation module'. It follows from § 3 that Γ can be represented as a group of endomorphisms on a module $\mathfrak{M}$ if and only if it has a representation $\mathfrak{G}$ of e-matrices for $\mathfrak{M}$.

Any representation $\mathfrak{G}$ is the direct sum of a representation $\mathfrak{G}_1$ all of whose matrices are a-matrices and a representation $\mathfrak{G}_0$ all of whose matrices are zero matrices. This follows from a Pierce-decomposition of the representation module. I shall assume from now on that $\mathfrak{G}$ is always a group of a-matrices.

The fundamental problem of this paper can then be formulated as follows: *Find the different, inequivalent representations of a finite group* Γ *as a group of a-matrices in a fixed ring* $\mathfrak{r}(p^r)$. Without loss of generality the representations may be assumed not to lie in $\mathfrak{r}(p^{r-1})$, which implies $p^{r-1}I \neq Z$.†

Notation. (1) h denotes the order of Γ and at the same time the corresponding element in $\mathfrak{r}(p^r)$; h' denotes the inverse of h in $\mathfrak{r}(p^r)$ if it exists.‡

(2) If $\mathfrak{S}$ is a set of e-matrices in $\mathfrak{r}(p^r)$, then $\mathfrak{S}$, considered mod $\mathfrak{p}^l$ $(l > 0)$, will be denoted by $\mathfrak{S}_{p^l}$, and the matrix representing the class of $S \pmod{\mathfrak{p}^l}$ will be denoted by S_{p^l}. The latter is a matrix in $\mathfrak{r}(p^l)$.

Definition 4.1. A representation of Γ in $\mathfrak{r}(p^r)$ is said to be of the *first kind* if $(p, h) = 1$. Otherwise the representation is said to be of the *second kind*.

All representations in what follows are assumed to be of the first kind.

If $\mathfrak{G}$ is a representation of Γ, then the closure $[\mathfrak{G}]$ of $\mathfrak{G}$ with respect

† More explicitly: If $\mathfrak{M}$ is the representation module, then $\exists\ u \in \mathfrak{M}$, $p^{r-1}u \neq \overline{0}$.

‡ It will be seen from the following remarks that h' may always be assumed to exist.

to addition and multiplication by $\mathfrak{r}(p^r)$ is a representation of the corresponding closure $[\Gamma]$ of Γ. Any property of invariance, reducibility, or decomposability of $\mathfrak{G}$ implies the corresponding property of $[\mathfrak{G}]$, and vice versa. We can therefore without further discussion operate either in $\mathfrak{G}$ or in $[\mathfrak{G}]$.

LEMMA 4.1. (1) *The sub-module* $\mathfrak{N}$ *of* $\mathfrak{M}$, *consisting of all elements of the form* $v = p^l u$ $(l > 0)$ *is invariant under any set* $\mathfrak{S}$ *of e-matrices.*

(2) *For every base of* $\mathfrak{M}$, $\mathfrak{N} = \mathfrak{p}^l\mathfrak{M}$.

(3) *The relation* $\qquad S \equiv S'\,(\mathfrak{p}^l)$

is equivalent to $\qquad S \equiv S' \quad (p^l I)$.

(4) $S \equiv S'$, $T \equiv T'$ $(\mathfrak{p}^l)$ *imply*

$$S+T \equiv S'+T', \qquad ST \equiv S'T'(\mathfrak{p}^l).\dagger$$

Proof. (1) follows from (3.3); (2) follows from the definition of $\mathfrak{N}$; (3) follows from the definition of the symbol $(\mathfrak{p}^l)$; (4) follows from (3).

LEMMA 4.2. *If* $\mathfrak{S}$ *is a set of e-matrices in* $\mathfrak{r}(p^r)$ *for the module* $\mathfrak{M}$, *then* $\mathfrak{S}_{p^l}$ *is a set of e-matrices for the difference module* $\mathfrak{M}-\mathfrak{p}^l\mathfrak{M}$. *If* $\mathfrak{S}$ *is a representation of the first kind, then so is* $\mathfrak{S}_{p^l}$.‡

Proof. By Lemma 4.1, $\mathfrak{p}^l\mathfrak{M}$ is invariant. Hence any $S \in \mathfrak{S}$ induces an endomorphism on $\mathfrak{M}-\mathfrak{p}^l\mathfrak{M}$; and two matrices S and S' induce the same endomorphism on $\mathfrak{M}-\mathfrak{p}^l\mathfrak{M}$ if and only if

$$S \equiv S'(\mathfrak{p}^l), \quad \text{i.e.} \quad S_{p^l} = S'_{p^l}.$$

If $\mathfrak{S}$ is a group, then, by Lemma 4.1 (4), so is $\mathfrak{S}_{p^l}$. If the order of $\mathfrak{S}$ is relatively prime to p, then so is the order of its quotient group $\mathfrak{S}_{p^l}$.

THEOREM 4.1. *If a representation* $\mathfrak{G}$ *in* $\mathfrak{r}(p^r)$ *of a group* Γ *is* $\mathfrak{a}$*-decomposable, then it is completely decomposable.*

The proof of this theorem is based on two lemmas.

LEMMA 4.3. *If* $\mathfrak{G}$ *is decomposable, then it is completely decomposable.*

LEMMA 4.4. *If* $\mathfrak{G}$ *is* $\mathfrak{p}^l$*-decomposable, then it is completely* $\mathfrak{p}^l$*-decomposable.*

Proof of Lemma 4.3. By Lemma 3.1 we may assume that

$$T(\gamma) = \begin{pmatrix} T_{11}(\gamma) & T_{12}(\gamma) \\ Z_{21} & T_{22}(\gamma) \end{pmatrix} \quad (\text{all } \gamma \in \Gamma). \tag{4.1}$$

Here, as in all similar expressions in what follows, $T_{11}(\gamma)$, $T_{22}(\gamma)$ are square matrices, and $T_{ik}(\gamma)$ and Z_{ik} the corresponding rectangular matrices.

† The case $l > r$ can obviously always be disregarded.

‡ Cf. Theorem 5.1.

From $$T(\gamma\delta) = T(\gamma)T(\delta) \quad (\text{all } \gamma, \delta \in \Gamma)$$
we obtain $$T_{ii}(\gamma\delta) = T_{ii}(\gamma)T_{ii}(\delta) \quad (i = 1, 2), \tag{4.2}$$
$$T_{12}(\gamma\delta) = T_{11}(\gamma)T_{12}(\delta)+T_{12}(\gamma)T_{22}(\delta). \tag{4.3}$$
Multiplying (4.3) from the right by $T_{22}(\delta^{-1})$ and summing over all $\delta \in \Gamma$, we get
$$\sum_\delta [T_{12}(\gamma\delta)T_{22}(\delta^{-1})] = T_{11}(\gamma) \sum_\delta [T_{12}(\delta)T_{22}(\delta^{-1})]+hT_{12}(\gamma). \tag{4.4}$$
Defining now R_{12} by
$$R_{12} = h' \sum_\delta T_{12}(\delta)T_{22}(\delta^{-1}), \tag{4.5}$$
we transform (4.4) into
$$hR_{12}T_{22}(\gamma) = hT_{11}(\gamma)R_{12}+hT_{12}(\gamma) \quad (\text{all } \gamma \in \Gamma). \tag{4.6}$$
We now define R by $$R = \begin{pmatrix} E_{11} & R_{12} \\ Z_{21} & E_{22} \end{pmatrix}, \tag{4.7}$$
where E_{11}, E_{22} are unit matrices of the same degrees as $T_{11}(\gamma)$, $T_{22}(\gamma)$ respectively. R is an a-matrix of the same type as $T(\gamma)$. Multiplying (4.6) by h' and using (4.7), we obtain
$$T(\gamma)R = R\begin{pmatrix} T_{11}(\gamma) & Z_{12} \\ Z_{21} & T_{22}(\gamma) \end{pmatrix} \quad (\text{all } \gamma \in \Gamma). \tag{4.8}$$
This is the required result.†

Proof of Lemma 4.4. By Lemma 3.1 we may assume that
$$T_{p^l}(\gamma) = \begin{pmatrix} T_{p^l11}(\gamma) & T_{p^l12}(\gamma) \\ Z_{21} & T_{p^l22}(\gamma) \end{pmatrix} \quad (\text{all } \gamma \in \Gamma). \tag{4.9}$$
By the proof of Lemma 4.3 there exists an a-matrix R of the same type as $T(\gamma)$ such that
$$T_{p^l}(\gamma)R_{p^l} = R_{p^l}\begin{pmatrix} T_{p^l11}(\gamma) & Z_{12} \\ Z_{21} & T_{p^l22}(\gamma) \end{pmatrix} \quad (\text{all } \gamma \in \Gamma). \tag{4.10}$$
This is equivalent to
$$T(\gamma)R \equiv R\begin{pmatrix} T_{11}(\gamma) & Z_{12} \\ Z_{21} & T_{22}(\gamma) \end{pmatrix} \quad (p^l I), \tag{4.11}$$
which proves Lemma 4.4.

Proof of Theorem 4.1. (1) By Lemma 3.1 we may assume that
$$T(\gamma) = \begin{pmatrix} T_{11}(\gamma) & T_{12}(\gamma) \\ T_{21}(\gamma) & T_{22}(\gamma) \end{pmatrix} \quad (\text{all } \gamma \in \Gamma), \tag{4.12}$$
where $$T_{21}(\gamma) \equiv Z_{21} \quad (\mathfrak{a}) \quad (\text{all } \gamma \in \Gamma), \tag{4.13}$$

† The proof is seen to be analogous to the corresponding proof for representation in fields.

$\mathfrak{a}$ is of the form $\{(p^{t_1}),..., (p^{t_m})\}$ $(t_i > 0)$. It follows that for $l = \min t_i > 0$

$$T_{21}(\gamma) \equiv Z_{21} \quad (p^l I) \quad (\text{all } \gamma \in \Gamma). \tag{4.14}$$

By Lemma 4.4 we may then also assume

$$T_{12}(\gamma) \equiv Z_{12} \quad (p^l I) \quad (\text{all } \gamma \in \Gamma). \tag{4.15}$$

I shall show that there exists a representation $\mathfrak{G}'$ with matrices $T'(\gamma)$ such that

$$\mathfrak{G}' \simeq \mathfrak{G} \tag{4.16}$$

and that

$$T'(\gamma) \equiv \begin{pmatrix} T'_{11}(\gamma) & Z_{12} \\ Z_{21} & T'_{22}(\gamma) \end{pmatrix} \quad (p^{2l} I) \quad (\text{all } \gamma \in \Gamma). \tag{4.17}$$

The theorem then follows by complete induction in n in the ideal $(p^{2^n l} I)$.

(2) It follows from (4.12), (4.14), and (4.15) that $T(\gamma\delta) = T(\gamma)T(\delta)$ implies, for all $\gamma, \delta \in \Gamma$,

$$T_{ii}(\gamma\delta) \equiv T_{ii}(\gamma)T_{ii}(\delta) \quad (p^{2l} I) \quad (i = 1, 2), \tag{4.18}$$

$$T_{12}(\gamma\delta) \equiv T_{11}(\gamma)T_{12}(\delta)+T_{12}(\gamma)T_{22}(\delta) \quad (p^{2l} I). \tag{4.19}$$

If we multiply (4.19) from the right by $T_{22}(\delta^{-1})$, sum over $\delta \in \Gamma$, and put

$$R_{12} = h' \sum_{\delta} T_{12}(\delta)T_{22}(\delta^{-1}),$$

we obtain, as above,

$$T_{11}(\gamma)R_{12}+T_{12}(\gamma) \equiv R_{12}T_{22}(\gamma) \quad (p^{2l} I) \quad (\text{all } \gamma \in \Gamma), \tag{4.20}$$

$$R_{12} \equiv Z_{12} \quad (p^l I). \tag{4.21}$$

In a similar way we find a matrix R_{21} such that

$$T_{22}(\gamma)R_{21}+T_{21}(\gamma) \equiv R_{21}T_{11}(\gamma) \quad (p^{2l} I) \quad (\text{all } \gamma \in \Gamma), \tag{4.20 a}$$

$$R_{21} \equiv Z_{21} \quad (p^l I). \tag{4.21 a}$$

Defining R by

$$R = \begin{pmatrix} E_{11} & R_{12} \\ R_{21} & E_{22} \end{pmatrix} \tag{4.22}$$

we see that R is an $\mathfrak{a}$-matrix of the same type as $T(\gamma)$, and that

$$T(\gamma)R \equiv R\begin{pmatrix} T_{11}(\gamma) & Z_{12} \\ Z_{21} & T_{22}(\gamma) \end{pmatrix} \quad (p^{2l} I) \quad (\text{all } \gamma \in \Gamma). \tag{4.23}$$

This shows that (4.16) and (4.17) follow from (4.14) and (4.15), which concludes the proof.

Corollary (1) *Any representation $\mathfrak{G}$ completely decomposes into not $\mathfrak{a}$-decomposable constituents.* (2) *The representation module of an indecomposable representation $\mathfrak{G}$ is of type $(p^s,..., p^s)$.*

The proof of (2) follows from Theorems 3.1 and 4.1.

(3) *If a representation* $\mathfrak{G}$ *decomposes into not* $\mathfrak{a}$-*decomposable constituents in two different ways:*

$$\mathfrak{G} = \mathfrak{G}_1 \dot{+} \mathfrak{G}_2 \dot{+} \dots \dot{+} \mathfrak{G}_n,$$

$$\mathfrak{G} = \mathfrak{G}'_1 \dot{+} \mathfrak{G}'_2 \dot{+} \dots \dot{+} \mathfrak{G}'_l,$$

then $l = n$, *and we can order the* $\mathfrak{G}'_i$ *in such a way that*

$$\mathfrak{G}_i \simeq \mathfrak{G}'_i.$$

Proof of (3). This follows from the general uniqueness theorem of decomposition for abelian groups with operators.

5. In this section a complete parallelism between representations in $\mathfrak{r}(p^r)$ and representations in the Galois field $\mathrm{GF}(p) = \mathfrak{r}(p)$ will be deduced.

THEOREM 5.1. *If* $\mathfrak{G}$ *is a representation in* $\mathfrak{r}(p^r)$, *then*

$$\mathfrak{G} \approx \mathfrak{G}_{p^l}, \quad 0 < l < r.$$

Proof. Evidently $\mathfrak{G} \sim \mathfrak{G}_{p^l}$. Let ι be the identity of Γ and suppose that

$$\exists\, T(\gamma) \in \mathfrak{G} \;\;.\;\; T(\gamma) \neq I, \qquad T(\gamma) \equiv I \quad (p^l I). \tag{5.1}$$

Then
$$T(\gamma) = I + p^l R(\gamma). \tag{5.2}$$

Hence
$$\{T(\gamma)\}^{p^k} = I \quad \text{for some } k > 0.$$

Hence
$$\gamma^{p^n} = \iota \quad \text{for some } n > 0. \tag{5.3}$$

But this contradicts the assumption that the representation is of the first kind. Therefore (5.1) cannot be true, and

$$\mathfrak{G} \approx \mathfrak{G}_{p^l}.$$

THEOREM 5.2. *A representation* $\mathfrak{G}$ *in* $\mathfrak{r}(p^r)$ *is not* $\mathfrak{a}$-*decomposable if and only if* $\mathfrak{G}_p$ *is indecomposable.*

Proof. By Theorem 4.1, $\mathfrak{a}$-decomposability of $\mathfrak{G}$ is equivalent to $\mathfrak{p}$-decomposability, and $\mathfrak{p}$-decomposability of $\mathfrak{G}$ is by definition equivalent to decomposability of $\mathfrak{G}_p$.

COROLLARY. *If* $\mathfrak{G}$ *completely decomposes into not* $\mathfrak{a}$-*decomposable constituents*

$$\mathfrak{G} = \mathfrak{G}_1 \dot{+} \mathfrak{G}_2 \dot{+} \dots \dot{+} \mathfrak{G}_n,$$

then $\mathfrak{G}_p$ *completely decomposes into indecomposable constituents*

$$\mathfrak{G}_p = (\mathfrak{G}_1)_p \dot{+} (\mathfrak{G}_2)_p \dot{+} \dots \dot{+} (\mathfrak{G}_n)_p.$$

If $\mathfrak{G}_p = \mathfrak{G}^*$ *completely decomposes into indecomposable constituents*

$$\mathfrak{G}^* = \mathfrak{G}^*_1 \dot{+} \mathfrak{G}^*_2 \dot{+} \dots \dot{+} \mathfrak{G}^*_n,$$

then $\mathfrak{G}$ *completely decomposes into not* $\mathfrak{a}$*-decomposable constituents*

$$\mathfrak{G} = \mathfrak{G}_1 \dot{+} \mathfrak{G}_2 \dot{+} \ldots \dot{+} \mathfrak{G}_n$$

such that $(\mathfrak{G}_i)_p = \mathfrak{G}_i^*$.

THEOREM 5.3. *Two not* $\mathfrak{a}$*-decomposable representations* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *in* $\mathfrak{r}(p^r)$ *are equivalent if and only if* $\mathfrak{G}_p$ *and* $\mathfrak{G}'_p$ *are equivalent.*

Proof. (1) Obviously it is sufficient to show that, for $r > 1$,

$$\mathfrak{G} \simeq \mathfrak{G}' \tag{5.4}$$

if and only if

$$\mathfrak{G}_{p^{r-1}} \simeq \mathfrak{G}'_{p^{r-1}}. \tag{5.5}$$

(2) Suppose that (5.4) holds. By Theorem 4.1, Corollary 2, both $\mathfrak{G}$ and $\mathfrak{G}'$ are of type $(p^r,\ldots,p^r)$. Hence there exists a constant a-matrix R of the same type such that, for $T(\gamma) \in \mathfrak{G}$, $T'(\gamma) \in \mathfrak{G}'$,

$$R^{-1}T(\gamma)R = T'(\gamma).$$

Hence

$$R^{-1}T(\gamma)R \equiv T'(\gamma) \quad (p^{r-1}I); \tag{5.6}$$

(5.6) however implies (5.5).

(3) Conversely suppose (5.5) to hold. Then there exists a constant a-matrix U in $\mathfrak{r}(p^{r-1})$, such that, for all $\gamma \in \Gamma$,

$$U^{-1}T_{p^{r-1}}(\gamma)U = T'_{p^{r-1}}(\gamma). \tag{5.7}$$

Any matrix V in $\mathfrak{r}(p^r)$ of the same degree as $\mathfrak{G}$ and $\mathfrak{G}'$ is also of the same type. Choosing V so that

$$V_{p^{r-1}} = U \tag{5.8}$$

and defining a representation $\mathfrak{G}_1$ by

$$T_1(\gamma) = V^{-1}T(\gamma)V, \tag{5.9}$$

we obtain from (5.7)

$$T_1(\gamma) = T'(\gamma) + p^{r-1}Q(\gamma) \quad (\text{all } \gamma \in \Gamma). \tag{5.10}$$

The theorem is then proved if it is shown that there exists a constant a-matrix R in $\mathfrak{r}(p^r)$ of the same degree as $\mathfrak{G}_1$ and $\mathfrak{G}'$ such that, for all $\gamma \in \Gamma$,

$$R^{-1}T_1(\gamma)R = T'(\gamma). \tag{5.11}$$

(4) Comparing the two equations

$$T'(\gamma\delta) = T'(\gamma)T'(\delta); \qquad T_1(\gamma\delta) = T_1(\gamma)T_1(\delta) \quad (\text{all } \gamma, \delta \in \Gamma),$$

we deduce from (5.10)

$$p^{r-1}Q(\gamma\delta) = p^{r-1}[Q(\gamma)T'(\delta) + T'(\gamma)Q(\delta)]. \tag{5.12}$$

Let now

$$W = h' \sum_{\delta} Q(\delta)T'(\delta^{-1}). \tag{5.13}$$

Multiplying (5.12) from the right by $T'(\delta^{-1})$ and summing over $\delta \in \Gamma$ we derive in the usual manner

$$p^{r-1}WT'(\gamma) = p^{r-1}[Q(\gamma) + T'(\gamma)W] \quad (\text{all } \gamma \in \Gamma). \tag{5.14}$$

Finally defining R by $$R = I+p^{r-1}W, \tag{5.15}$$
we obtain from (5.10) and (5.14)
$$T_1(\gamma)R = RT'(\gamma). \tag{5.16}$$
By (5.15), $$|R| \not\equiv 0 \quad (p).$$
Hence (5.16) implies (5.11).

COROLLARY (1) *Theorem 5.3 is also true if $\mathfrak{G}$ and $\mathfrak{G}'$ are $\mathfrak{a}$-decomposable representations of type $(p^r,..., p^r)$.*

Proof. The only use that was made of the hypothesis that $\mathfrak{G}$ and $\mathfrak{G}'$ be not $\mathfrak{a}$-decomposable was to deduce the fact that they are of type $(p^r,..., p^r)$.

The theorem is, however, not true in general if this condition is not satisfied. For example, let Γ consist of the identity only. Let $\mathfrak{G}_1$ be its representation on a module of type (p^2, p^2) and $\mathfrak{G}_2$ its representation on a module of type (p^2, p). Then $\mathfrak{G}_1 \not\simeq \mathfrak{G}_2$, but $(\mathfrak{G}_1)_p \simeq (\mathfrak{G}_2)_p$.

It is possible to find more general criteria than the above. This however is unnecessary for our purpose.

THEOREM 5.4. *If a finite group Γ has exactly l inequivalent, indecomposable representations $\mathfrak{G}_1^*,..., \mathfrak{G}_l^*$ in $\mathrm{GF}(p)$, then it has exactly l inequivalent, not $\mathfrak{a}$-decomposable representations $\mathfrak{G}_1,..., \mathfrak{G}_l$ in $\mathfrak{r}(p^r)$ where the $\mathfrak{G}_i$ can be ordered in such a way, that*
$$(\mathfrak{G}_i)_p = \mathfrak{G}_i^* \quad (i = 1, 2,..., l). \tag{5.17}$$

Proof. So far I have shown that to each $\mathfrak{G}_i$ there corresponds a $\mathfrak{G}_i^*$ such that (5.17) holds, and that $\mathfrak{G}_i \not\simeq \mathfrak{G}_k$ if and only if
$$(\mathfrak{G}_i)_p \not\simeq (\mathfrak{G}_k)_p.$$
It remains to be proved that for each $\mathfrak{G}_i^*$ there actually exists a $\mathfrak{G}_i$ satisfying (5.17).

For this purpose I introduce the regular representation $\overline{\mathfrak{G}}$ of Γ in $\mathfrak{r}(p^r)$. We consider $[\Gamma]$ as a module whose general element is $\beta = \sum a(\delta)\delta$, where $a(\delta) \in \mathfrak{r}(p^r)$ and the summation extends over all $\delta \in \Gamma$. $\overline{\mathfrak{G}}$ is then defined by $$\overline{T}(\gamma)\delta = \gamma\delta \quad (\text{all } \gamma, \delta \in \Gamma). \tag{5.18}$$

$\overline{\mathfrak{G}}_p$ is evidently the regular representation of Γ in $\mathrm{GF}(p)$. Its indecomposable constituents are exactly $\mathfrak{G}_1^*,..., \mathfrak{G}_l^*$. By Theorem 5.2 and Theorem 5.3, $\overline{\mathfrak{G}}$ contains then exactly l inequivalent, not $\mathfrak{a}$-decomposable representations $\mathfrak{G}_1,..., \mathfrak{G}_l$, such that (5.17) is satisfied. Finally, every not $\mathfrak{a}$-decomposable representation $\mathfrak{G}$ of Γ in $\mathfrak{r}(p^r)$ satisfies the relation
$$\mathfrak{G}_p \simeq \mathfrak{G}_i^* \quad (\text{some } i),$$
and hence $$\mathfrak{G} \simeq \mathfrak{G}_i.$$

6. In this section I shall discuss the reducibility properties of a not $\mathfrak{a}$-decomposable representation and the structure of its commutator ring.

THEOREM 6.1. *If $\mathfrak{M}$ is the representation module of a not $\mathfrak{a}$-decomposable representation $\mathfrak{G}$ in $\mathfrak{r}(p^r)$, then a sub-module $\mathfrak{N}$ of $\mathfrak{M}$ is invariant under $\mathfrak{G}$ if and only if*

$$\mathfrak{N} = \mathfrak{p}^l\mathfrak{M}.$$

Proof. By Lemma 4.1, $\mathfrak{p}^l\mathfrak{M}$ is invariant. Suppose, conversely, that $\mathfrak{N}$ is invariant. Let $(u_1,\ldots, u_m)$ be a base of $\mathfrak{M}$ such that $(p^{t_1}u_1,\ldots, p^{t_m}u_m)$ generate $\mathfrak{N}$, and let $t_i \leqslant t_{i+1}$. Suppose that

$$t_1 = \ldots = t_k < t_{k+1}. \tag{6.1}$$

Then $(u_1,\ldots, u_k)$ generate a module $\mathfrak{M}_1$, $(u_{k+1},\ldots, u_m)$ generate a module $\mathfrak{M}_2$, and $\mathfrak{M} = \mathfrak{M}_1 \dotplus \mathfrak{M}_2$. By hypothesis $\exists\ T \in \mathfrak{G}$, $v_1 \in \mathfrak{M}_1$, $v_1 \notin \mathfrak{p}\mathfrak{M}_1$ such that

$$Tv_1 = v_1'+v_2', \qquad v_1' \in \mathfrak{M}_1, \qquad v_2' \in \mathfrak{M}_2, \qquad v_2' \notin \mathfrak{p}\mathfrak{M}_2. \tag{6.2}$$

Then
$$Tp^{t_1}v_1 = p^{t_1}v_1'+p^{t_1}v_2'. \tag{6.3}$$

By (6.1),
$$\bar{0} \neq p^{t_1}v_1 \in \mathfrak{N}. \tag{6.4}$$

By (6.1), (6.2),
$$p^{t_1}v_1' \in \mathfrak{N}, \qquad p^{t_1}v_2' \notin \mathfrak{N}.$$

Hence
$$Tp^{t_1}v_1 \notin \mathfrak{N}. \tag{6.5}$$

Comparison of (6.4) and (6.5) shows that $\mathfrak{N}$ is not invariant, contrary to assumption. Thus (6.1) leads to a contradiction. Hence

$$\mathfrak{N} = \mathfrak{p}^l\mathfrak{M}.$$

The following theorem is a generalization of Schur's lemma.

THEOREM 6.2. (1) *If $\mathfrak{G}$ and $\mathfrak{G}'$ are two inequivalent, not $\mathfrak{a}$-decomposable representations of a group Γ in $\mathfrak{r}(p^r)$, and if there exists a constant matrix R such that, for $T(\gamma) \in \mathfrak{G}$, $T'(\gamma) \in \mathfrak{G}'$,*

$$T(\gamma)R = RT'(\gamma) \quad (\text{all } \gamma \in \Gamma), \tag{6.6}$$

then
$$R = Z. \tag{6.7}$$

(2) *If $\mathfrak{G}$ is a not $\mathfrak{a}$-decomposable representation in $\mathfrak{r}(p^r)$, and a constant matrix Q satisfies the relations*

$$T(\gamma)Q = QT(\gamma) \quad (\textit{all } \gamma \in \Gamma), \tag{6.8}$$

$$Q \neq Z, \tag{6.9}$$

then
$$Q = p^tQ_1, \qquad |Q_1| \not\equiv 0 \ (p) \quad (t < r). \tag{6.10}$$

Proof. (1) Suppose that R satisfies (6.6) but not (6.7). Then

$$R = p^tR_1, \qquad R_1 \not\equiv Z \ (pI) \quad (t < r). \tag{6.11}$$

Also
$$T(\gamma)R_1 \equiv R_1\, T'(\gamma) \ (pI) \quad (\text{all } \gamma \in \Gamma). \tag{6.12}$$

This however contradicts Schur's lemma for matrices in GF(p). Therefore (6.11) cannot hold, and $R = Z$.

(2) Suppose that Q satisfies (6.8) and (6.9). Then

$$Q = p^t Q_1, \qquad Q_1 \not\equiv Z \ (pI) \quad (t < r). \tag{6.13}$$

Also, by (6.8),

$$T(\gamma)Q_1 \equiv Q_1 T(\gamma) \ (pI) \quad (\text{all } \gamma \in \Gamma). \tag{6.14}$$

Hence, by Schur's lemma for matrices in GF(p),

$$|Q_1| \not\equiv 0 \quad (p).$$

THEOREM 6.3. *If $\mathfrak{G}$ is a not $\mathfrak{a}$-decomposable representation in $\mathfrak{r}(p^r)$, and if, for all $T(\gamma) \in \mathfrak{G}$,*

$$T(\gamma)Q = QT(\gamma), \tag{6.15}$$

$$Q \equiv Z \quad (p^t I) \quad (t < r), \tag{6.16}$$

then $\exists$ Q_1 such that

$$Q = p^t Q_1, \tag{6.17}$$

$$T(\gamma)Q_1 = Q_1 T(\gamma) \ (\text{all } \gamma \in \Gamma). \tag{6.18}$$

Proof. (1) I shall prove that, if, for all $T(\gamma) \in \mathfrak{G}$,

$$T(\gamma)Q' \equiv Q'T(\gamma) \ (p^{r-1}I), \tag{6.19}$$

then $\exists$ Q_1 such that

$$Q_1 \equiv Q' \ (p^{r-1}I), \tag{6.20}$$

$$T(\gamma)Q_1 = Q_1 T(\gamma) \quad (\text{all } \gamma \in \Gamma). \tag{6.21}$$

Assuming that this is true, and that, in (6.16), $t = 1$, then (6.15) and (6.16) imply $Q = pQ'$, where Q' satisfies (6.19). Then (6.17) follows from (6.20), and (6.18) is identical with (6.21). For $t > 1$ the theorem follows by complete induction.

(2) From (6.19) we obtain

$$T(\gamma)Q' = Q'T(\gamma) + p^{r-1}V(\gamma). \tag{6.22}$$

The proof proceeds now along similar lines to the proof of Theorem 5.3. We deduce

$$p^{r-1}V(\gamma\delta) = p^{r-1}[V(\gamma)T(\delta) + T(\gamma)V(\delta)]. \tag{6.23}$$

Defining Q_1 by

$$Q_1 = Q' + p^{r-1} \sum_{\delta} V(\delta)T(\delta^{-1}), \tag{6.24}$$

we easily verify in the same way as in earlier proofs that Q_1 satisfies both (6.20) and (6.21).

We have determined the structure of all representations of a finite group of order h as a group of automorphisms on an abelian group of order n, whenever $(h, n) = 1$. The results are, however, not true if $(h, n) \neq 1$.

The method used in establishing the results of this paper is not the only possible one; but it seems to be the most elementary one, in so far as it does not assume any knowledge of general ring-theory.

REFERENCES

1. A. Speiser, *Die Theorie der Gruppen von endlicher Ordnung* (Berlin, 1937).
2. B. L. Van der Waerden, *Moderne Algebra* (Berlin, 1937).
3. K. Shoda, 'Über die Automorphismen einer endlichen Abelschen Gruppe', *Math. Annalen* 100 (1928), 674–86.

COVERING THEOREMS FOR GROUPS

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[Received 28 September 1949]

1. Introduction

THIS is a note on a paper with the above title by O. Taussky and J. Todd (**1**). As far as possible the same notation will be used.

The first problem in (**1**) may be stated as follows: *We are given the set G of p^n vectors, each with n coordinates, and such that each coordinate can take precisely p values. We now wish to determine a sub-set H of G (as small as possible) which has the following property: any vector of G has at least $n-1$ coordinates in common with some vector of H.* In this paper I confine attention to the case when p is a prime.

Clearly, given any vector of H, the number of vectors of G which differ from it in not more than one coordinate is $\nu = n(p-1)+1$, so that the number of vectors in H must be at least p^n/ν.

Notice first that, as in (**1**), G can be represented as an abelian group, which I write additively, with n generators g_s and relations $pg_s = 0$ ($s = 1,..., n$). Under this representation H may or may not be a sub-group of G. We define the integer $\sigma(n) = \sigma(n,p)$ as the smallest possible number of vectors in the sub-set H of G, and $\sigma^*(n)$ as the smallest number subject to the further restriction that H is to be a sub-group of the group G. Clearly

$$p^n/\nu \leqslant \sigma(n) \leqslant \sigma^*(n) \leqslant p^n. \tag{1.1}$$

The second problem in (**1**) is to determine the cases when

$$\sigma(n,p) = \sigma^*(n,p).$$

2. The case $n = (p^r-1)/(p-1)$

To avoid a trivial case not subject to the general treatment I assume $r \geqslant 2$. It will be found that in this case $\sigma(n) = \sigma^*(n) = p^{n-r}$.

Instead of taking the symbols g_s ($s = 1,..., n$) as generators of the group G, I adopt a different notation and use the symbols a_{ij} and b_i (where $i = 0,..., r-1$; $j = 1,..., p^i-1$) with relations $pa_{ij} = pb_i = 0$. It should be noted that there are no generators of the type a_{0j}, and that the total number of generators is

$$\sum_{i=0}^{r-1} (p^i-1)+r = (p^r-1)/(p-1) = n$$

as required.

Now, for any given values of i and j with $0 \leqslant i \leqslant r-1$, $1 \leqslant j \leqslant p^i-1$, we define integers j_u ($u = 0,..., r-1$) by expressing j in the p-ary scale as

$$j = \sum_{u=0}^{r-1} j_u p^u \quad (0 \leqslant j_u \leqslant p-1). \tag{2.1}$$

Thus $j_u = 0$ for $u \geqslant i$.

Next we define integers $\lambda_{ij}^{(u)}$ ($i, u = 0,..., r-1$; $j = 1,..., p^i-1$) by writing

$$\lambda_{ij}^{(u)} = j_u \quad (u \neq i), \qquad \lambda_{ij}^{(i)} = 1. \tag{2.2}$$

Finally we define the elements c_{ij} ($i = 1,..., r-1$; $j = 1,..., p^i-1$) of the group G by the equations:

$$c_{ij} = \sum_{u=0}^{r-1} \lambda_{ij}^{(u)} b_u + a_{ij} \tag{2.3}$$

and note that admitting the value $i = 0$ would have given no significant contribution.

The sub-group of G generated by the elements c_{ij} can be taken as the required set H.

I first show that for any element g of G there exists an element h of H which, when expressed as a sum of generators of G, differs (mod p) from g in at most one coefficient. In fact let

$$g = \sum_{ij} l_{ij} a_{ij} + \sum_i m_i b_i,$$

where the coefficients l_{ij}, m_i are integers, and consider the integers

$$M_u = m_u - \sum_{ij} \lambda_{ij}^{(u)} l_{ij} \quad (u = 0,..., r-1), \tag{2.4}$$

where the values of $\lambda_{ij}^{(u)}$ are given by (2.2). Two cases now arise.

First. If not more than one of the M_u satisfies the condition

$$M_u \not\equiv 0 \pmod p,$$

put

$$h = \sum_{ij} l_{ij} c_{ij} \in H.$$

We easily verify by (2.3) and (2.4) that h has the required property.

Second. If at least two of the M_u satisfy the condition $M_u \not\equiv 0 \pmod p$, let w be the greatest value of u for which $M_u \not\equiv 0 \pmod p$. Thus $M_u \equiv 0 \pmod p$ $(u > w)$; however $M_w \not\equiv 0 \pmod p$, and there is at least one value of $u < w$ for which

$$M_u \not\equiv 0 \pmod p. \tag{2.5}$$

Using the fact that p is a prime we now define the integers μ_u for $u = 0,..., r-1$ by

$$0 \leqslant \mu_u \leqslant p-1, \qquad \mu_u M_w \equiv M_u \pmod p \quad (u \neq w), \qquad \mu_w = 0$$

and put

$$\mu = \sum_{u=0}^{r-1} \mu_u p^u.$$

Hence, by (2.5), $1 \leqslant \mu \leqslant p^w - 1$. By (2.1) and (2.2),

$$\lambda_{w\mu}^{(u)} = \mu_u \quad (u \neq w), \qquad \lambda_{w\mu}^{(w)} = 1,$$

so that in any case

$$M_u \equiv \lambda_{w\mu}^{(u)} M_w \pmod p. \tag{2.6}$$

Now put

$$h = \sum_{ij} l_{ij} c_{ij} + M_w c_{w\mu} \in H.$$

Then by (2.3), (2.4), (2.6) it follows that the coefficients of h agree $(\operatorname{mod} p)$ with those of g except for the coefficient of $a_{w\mu}$. Thus h has the required property.

The number of elements c_{ij} (like that of the symbols a_{ij}) is $n-r$. Thus H has p^{n-r} elements and $\sigma^*(n) \leqslant p^{n-r}$. Since in this case

$$\nu = n(p-1)+1 = p^r,$$

combining this inequality with (1.1) gives:

If $\qquad n = (p^r-1)/(p-1) \quad$ *then* $\quad \sigma(n) = \sigma^*(n) = p^{n-r}$. (2.7)

3. General values of n. An exact formula for $\sigma^*(n)$

If G is the direct sum G_1+G_2 and H_1 satisfies the required condition relative to G_1, then H_1+G_2 does so relative to G. In particular, if G_1, G_2 have respectively n_0, m generators, all of order p, we find

$$\sigma^*(n_0+m) \leqslant p^m \sigma^*(n_0). \tag{3.1}$$

Given n, let the integer r be determined by

$$p^r \leqslant n(p-1)+1 < p^{r+1}, \tag{3.2}$$

so that in a well-known notation $r = [\log \nu / \log p]$, and put

$$n_0 = (p^r-1)/(p-1), \qquad n = n_0+m.$$

By (3.1) and (2.7), $\qquad \sigma^*(n) \leqslant p^m \sigma^*(n_0) = p^{n-r}$

and, by (1.1) and (3.2),

$$p^{n-r-1} < p^n/\nu \leqslant \sigma(n) \leqslant \sigma^*(n) \leqslant p^{n-r}. \tag{3.3}$$

Now the order of a sub-group of G must be a factor of p^n; hence $\sigma^*(n)$ must be a power of the prime p. Hence, by (3.3),

$$\sigma^*(n) = p^{n-r} \tag{3.4}$$

for all n, where r is determined by (3.2).

4. The case $n = 2n_0+m$, $n_0 = (p^r-1)/(p-1)$, $1 \leqslant m \leqslant (p-2)n_0$

Using the notation of § 3, let

$$G = G_1+G_1'+G_2 \quad \text{where} \quad G_1' \approx G_1$$

and let

$$\vartheta : G_1' \approx G_1$$

be an isomorphism which sends generators on to generators. If $g' \in G_1'$

then $\vartheta g' \in G_1$. We find that a set H with the required property is given by all elements except 0 of the form

$$(h+\vartheta g')+g'+\bar{g} \in G \quad (h \in H_1,\ g' \in G_1',\ \bar{g} \in G_2)$$

and the number of elements in H is

$$p^{n_0-r}p^{n_0}p^m-1 = p^{n-r}-1.$$

Hence $$\sigma(n) \leqslant p^{n-r}-1,$$

but, by (3.4), $\sigma^*(n) = p^{n-r}$ so that in this case

$$\sigma(n) < \sigma^*(n). \tag{4.1}$$

5. The case $p = 2$, $n = 2^{r+1}-2$, $r \geqslant 2$

Let $G = G_1+G_2+K$, where G_1 has 2^r-1 generators

$$a_{11}, a_{21},..., b_0, b_1,..., b_{r-1},$$

G_2 has 2^r-4 generators, and K has three generators k_0, k_1, k_2, all of order 2.

Let $\hat{G}_1$ be the sub-group of G_1 generated by all its generators except a_{11}, b_0, b_1, and let $\vartheta: G_2 \approx \hat{G}_1$ be an isomorphism which sends generators on to generators. Let $f: K \to G_1$ be a mapping defined by

$$fk_0 = a_{11}, \qquad f(k_0+k_2) = b_0, \qquad f(k_1+k_2) = b_1,$$
$$fk = 0 \text{ for all other elements } k \text{ of } K.$$

By (2.3) the set of elements of $G = G_1+G_2+K$ defined by

$$(h+\vartheta\bar{g}+fk)+\bar{g}+k \quad (h \in H_1,\ \bar{g} \in G_2,\ k \in K)$$

contains 2^{n-r} elements including c_{11} and k_2. Excluding these and substituting the element a_{11} gives a set H which will be found to have the required property. Since by (3.4) $\sigma^*(n) = 2^{n-r}$, it follows that in this case

$$\sigma(n) < \sigma^*(n). \tag{5.1}$$

6. The second problem in (1)

From (2.7), (4.1), (5.1) we see that, if p is a prime, there is an infinity of values of n which provide solutions of the second problem in (**1**), and an infinity of cases of failure of the property.

REFERENCE

1. Olga Taussky and John Todd: 'Covering theorems for groups', *Ann. Soc. Polonaise de Math.* 21 (1948) 303–5.

ON LINEAR FUNCTIONALS IN SPACES OF CONDITIONALLY INTEGRABLE FUNCTIONS

By W. L. C. SARGENT (*London*)

[Received 5 October 1949]

1. Introduction

LET (EBV) denote the space of functions equivalent to functions of bounded variation in $(0, 1)$ and let (I) denote any one of the following three normalized vector spaces (the same space throughout): (i) the space of functions $x(t)$ integrable in the Cauchy–Lebesgue sense in $(0, 1)$, (ii) the space of functions integrable in the special Denjoy sense in $(0, 1)$, (iii) the space of functions integrable in the general Denjoy sense in $(0, 1)$, the norm in (I) being defined by the equation

$$||x|| = \overline{\underset{0\leqslant t\leqslant 1}{\text{bound}}}\left|\int_0^t x(u)\,du\right|.$$

Then it follows from a result* obtained recently that in order that $x(t)k(t) \in (I)$ whenever $x(t) \in (I)$, it is necessary and sufficient that $k(t) \in (EBV)$. Moreover, by arguments due to Alexiewicz,† the general linear functional in (I) is given by an equation of the form

$$f(x) = \int_0^1 x(t)k(t)\,dt,$$

where $k(t) \in (EBV)$. In this paper I find necessary and sufficient conditions for a sequence of linear functionals $\left\{\int_0^1 x(t)k_n(t)\,dt\right\}$ to be (i) bounded, (ii) convergent, in (I).

2. Notation

The notation is in general that given in Banach (**5**) and only additional notation will be defined here.

In addition to the spaces (I) and (EBV) which have already been defined, we shall be concerned with the following spaces: (i) the B-space (C_0) consisting of functions $y(t)$ which are continuous in the closed interval $[0, 1]$ and vanish at the origin, with norm

$$||y|| = \overline{\underset{0<t<1}{\text{bound}}}\,|y(t)|;$$

* W. L. C. Sargent (**10**), 29 and 32–4; necessity follows from the result obtained here, and it is well known that the condition is sufficient.

† A. Alexiewicz (**2**), 290; the space (iii) is considered here.

(ii) the vector sub-space (CI) of (C_0) consisting of functions $y(t)$ of the form $\int_0^t x(u)\,du$, where $x(t) \in (I)$, the norm being defined in the same way as in (C_0); (iii) the sub-space (BV) of (EBV) consisting of functions of bounded variation in $[0,1]$; (iv) the sub-space (BVN) of (BV) consisting of functions of the space (BV) which have no external saltus in $[0,1]$.*

If $k_1(t)$ and $k_2(t)$ are two equivalent functions of the space (EBV), I shall write $k_1(t) \cong k_2(t)$. If $k(t) \in (EBV)$, I shall define $V_e(k)$, the *essential variation* of $k(t)$ in $(0,1)$, to be the lower bound of $\int_0^1 |d\theta(t)|$ for all functions $\theta(t)$ such that $\theta(t) \in (BV)$ and $\theta(t) \cong k(t)$.

It should be observed that, if $k(t) \in (EBV)$, then the essential one-sided limits† of $k(t)$ exist at all points of $[0,1]$ and

$$\operatorname*{ess-lim}_{h\to+0} k(t+h) = \theta(t+0) \quad (0 \leqslant t < 1),$$

$$\operatorname*{ess-lim}_{h\to+0} k(t-h) = \theta(t-0) \quad (0 < t \leqslant 1),$$

where $\theta(t) \in (BV)$ and $\theta(t) \cong k(t)$. It follows that

$$V_e(k) = \int_0^1 |d\rho(t)|, \tag{2.1}$$

where $\rho(t) \in (BVN)$ and $\rho(t) \cong k(t)$.

3. Linear functionals in the spaces (C_0) and (CI)

We first observe that any function of the space (CI) has a finite derivative in a sub-set of $(0,1)$ of positive measure, and hence‡ that (CI) is of the first category in (C_0). Further, (CI) is dense in (C_0), and hence§ (CI) is also of the first category in itself.

Since (C_0) and (CI) are vector sub-spaces of the space (C), the general

* Thus (BVN) consists of functions $\theta(t)$ of the space (BV) for which $\theta(t)$ lies between $\theta(t+0)$ and $\theta(t-0)$ inclusive, whenever $0 < t < 1$, while $\theta(0) = \theta(+0)$ and $\theta(1) = \theta(1-0)$; cf. C. R. Adams and A. P. Morse (**1**), 195.

† It is to be understood that only a right-hand limit is considered at $t = 0$ and a left-hand limit at $t = 1$.

‡ Banach [(**4**), 175] has shown that the sub-set of (C) consisting of functions which have finite right-hand derivates at at least one point t $(0 \leqslant t < 1)$ is of the first category in (C), and it is easily seen that the same result holds with (C) replaced by (C_0).

§ Cf. C. R. Adams and A. P. Morse (**1**), 201.

linear functional in either of these spaces is of the form

$$\int_0^1 y(t)\,d\theta(t),$$

where $\theta(t) \in (BV)$. We next determine the norm of this functional, $\theta(t)$ denoting any function of the space (BV).

THEOREM 1. *If* $\theta(t) \in (BV)$ *and*

$$F(y) = \int_0^1 y(t)\,d\theta(t), \tag{3.1}$$

*then** $$||F||_{(CI)} = ||F||_{(C_0)} = V_e(\theta)+|\theta(1)-\theta(1-0)|.$$

Since (CI) is dense in (C_0), the two norms $||F||_{(CI)}$, $||F||_{(C_0)}$ are equal; their common value will be denoted by $||F||$.

Since (C_0) is a vector sub-space of (C), $F(y)$ can be expressed in the form

$$F(y) = \int_0^1 y(t)\,dg(t), \tag{3.2}$$

where $g(t) \in (BV)$ and $\int_0^1 |dg(t)| = ||F||$. Further, by subtracting a constant from $g(t)$ if necessary, we may suppose that $g(1) = \theta(1)$.

Let $h(t)$ denote any function such that $h(t) \in (BV)$, $h(t) \simeq g(t)$, and $h(1) = g(1)$. Then,† whenever $y(t) \in (C_0)$,

$$F(y) = \int_0^1 y(t)\,dh(t),$$

and hence, by familiar arguments, $||F|| \leqslant \int_0^1 |dh(t)|$. It follows that

$$\int_0^1 |dg(t)| \leqslant \int_0^1 |dh(t)|,$$

and hence that $g(t)$ can have no external saltus for $0 < t < 1$, while $g(+0) = g(0)$.

If we equate the values of $F(y)$ given by (3.1) and (3.2) and then integrate by parts, we find that, whenever $y(t) \in (C_0)$,

$$\int_0^1 \{\theta(t)-g(t)\}\,dy(t) = 0.$$

* The corresponding result

$$||F||_{(C)} = V_e(\theta)+|\theta(1)-\theta(1-0)|+|\theta(+0)-\theta(0)|$$

is implicit in the work of F. Riesz (**9**), 977; cf. also H. Hahn (**7**), 86.

† Cf. D. V. Widder (**11**), 12–13; since $y(0) = 0$, the condition $h(0) = g(0)$ may be dispensed with.

Taking $y(t) = \min(t, u)$, it follows that

$$\int_0^u \{\theta(t)-g(t)\}\,dt = 0 \quad (0 \leqslant u \leqslant 1),$$

and hence that $g(t) \simeq \theta(t)$. In view of the definition and properties of $g(t)$, it then follows that

$$\int_0^1 |dg(t)| = V_e(\theta)+|\theta(1)-\theta(1-0)|,$$

giving the result required.

We now obtain two theorems concerning a sequence of linear functionals. I denote by $F_n(y)$ a functional linear in (C_0); in view of the Hahn–Banach extension theorem, $F_n(y)$ may be regarded as an arbitrary linear functional defined in (CI) and extended to (C_0). It has already been pointed out that the norms in (CI) and (C_0) are equal; their common value will be denoted by $||F_n||$.

THEOREM 2. *In order that*

$$\overline{\lim_{n\to\infty}} |F_n(y)| < \infty \tag{3.3}$$

whenever $y(t) \in (C_0)$, *it is necessary and sufficient that* (3.3) *should hold for all* $y(t)$ *of* (CI).*

In order to obtain Theorem 2, it will be sufficient to suppose that

$$\overline{\lim_{n\to\infty}} F_n(\alpha) = \infty, \tag{3.4}$$

where $\alpha(t) \in (C_0)$, and show that there is a function $\beta(t)$ of (CI) such that

$$\overline{\lim_{n\to\infty}} F_n(\beta) = \infty. \tag{3.5}$$

By arguments used in Theorem 1, we may suppose that

$$F_n(y) = \int_0^1 y(t)\,d\sigma_n(t),$$

where $\sigma_n(t) \in (BV)$, $\sigma_n(+0) = \sigma_n(0)$, and $\sigma_n(t)$ has no external saltus for $0 < t < 1$. Further, by subtracting a constant from $\sigma_n(t)$ if necessary, we may suppose that $\sigma_n(1) = 0$. For convenience I shall also suppose that $\sigma_n(t) = \sigma_n(0)$ for $t < 0$ and $\sigma_n(t) = \sigma_n(1)$ for $t > 1$. I shall write

$$M_n = \overline{\underset{0<t<1}{\mathrm{bound}}}\, |\sigma_n(t)|.$$

* By a known result [cf. S. Banach (**5**), 80] the set H of functions $y(t)$ of (C_0) for which (3.3) holds is either of the first category in (C_0) or identical with (C_0); Theorem 2 shows that, although (CI) is of the first category in (C_0), $(CI) \subset H$ implies $H = (C_0)$.

We first consider the case when $\overline{\lim}_{n\to\infty} M_n = \infty$. In view of the definition of $\sigma_n(t)$, M_n is the essential upper bound of $|\sigma_n(t)|$ in $(0, 1)$. It therefore follows from a result due to Lebesgue* that

$$\overline{\lim}_{n\to\infty} \int_0^1 x(t)\sigma_n(t)\,dt = \infty$$

for at least one function $x(t)$ integrable in the Lebesgue sense in $(0,1)$; since $\sigma_n(1) = 0$, integration by parts then gives

$$\overline{\lim}_{n\to\infty} \int_0^1 \beta(t)\,d\sigma_n(t) = \infty,$$

where $\beta(t) = -\int_0^t x(u)\,du$ for $0 \leqslant t \leqslant 1$, hence $\beta(t) \in (CI)$.

It remains to consider the case when there is a constant M such that

$$M_n \leqslant M \quad (n = 1, 2, 3,...). \tag{3.6}$$

I shall call a point ξ of $[0, 1]$ *regular* if

$$\overline{\lim}_{n\to\infty} \int_{\xi-h}^{\xi+k} \alpha(t)\,d\sigma_n(t) < \infty$$

for all sufficiently small positive values of h and k. In view of (3.4), it follows from the Heine–Borel theorem that there is at least one point c of $[0, 1]$ which is not regular. I shall denote by E the set of points p of $(c, 1)$ for which

$$\overline{\lim}_{n\to\infty} \int_c^p \alpha(t)\,d\sigma_n(t) = \infty,$$

and shall suppose that c is a limit point of E; otherwise c is a limit point of the corresponding set of points of $(0, c)$ and the procedure is similar.

I first show that, if K and ν are arbitrary positive numbers and if $q \in E$, then there is a point p of E $(c < p < \frac{1}{2}q + \frac{1}{2}c)$ and a positive integer μ $(> \nu)$ such that

$$\left|\int_c^p \alpha(t)\,d\sigma_\mu(t)\right| < 2M||\alpha||+1, \qquad \int_p^q \alpha(t)\,d\sigma_\mu(t) > K+1.$$

Since $q \in E$, we can find μ $(> \nu)$ such that

$$\int_c^q \alpha(t)\,d\sigma_\mu(t) > K+2M||\alpha||+2.$$

Since†
$$\lim_{u\to c+0} \int_c^u \alpha(t)\,d\sigma_\mu(t) = \alpha(c)\{\sigma_\mu(c+0)-\sigma_\mu(c)\},$$

* H. Lebesgue (**8**), 53; cf. also H. Hahn (**7**), 40.

† Cf. D. V. Widder (**11**), 9.

we can then find a point p of E so that the required conditions are satisfied.

Having found μ and p, we can clearly approximate to $\alpha(t)$ in $[p,q]$ by a function $\beta(t)$, absolutely continuous in $[p,q]$ and such that $\beta(p) = \alpha(p)$, $\beta(q) = \alpha(q)$,

$$|\beta(t)-\alpha(t)| < \epsilon \quad (p < t < q),$$

and

$$\int_p^q \beta(t)\, d\sigma_\mu(t) > K,$$

ϵ being an arbitrary positive number.

It follows by induction that, corresponding to any sequences $\{K_r\}$, $\{\epsilon_r\}$ of positive numbers, there is a decreasing sequence $\{p_r\}$ of points of E, with limit c, an increasing sequence $\{n_r\}$ of positive integers such that

$$\left|\int_c^{p_{r+1}} \alpha(t)\, d\sigma_{n_r}(t)\right| < 2M||\alpha||+1 \tag{3.7}$$

and a function $\beta(t)$, absolutely continuous in each interval $[p_{r+1}, p_r]$ and such that, for every positive integer r,

$$\beta(p_r) = \alpha(p_r), \tag{3.8}$$

$$|\beta(t)-\alpha(t)| < \epsilon_r \quad (p_{r+1} < t < p_r), \tag{3.9}$$

$$\int_{p_{r+1}}^{p_r} \beta(t)\, d\sigma_{n_r}(t) > K_r. \tag{3.10}$$

In the definition by induction we may clearly define K_r and ϵ_r after $\beta(t)$ has been defined in $[p_r, p_1]$. Writing

$$A_r = \int_{p_r}^{p_1} |\beta'(t)|\, dt, \qquad V_r = \int_c^{p_r} |d\sigma_{n_{r-1}}(t)| \quad (r > 1),$$

I shall take

$$K_r = 6M||\alpha||+MA_r+r \tag{3.11}$$

and,* in the case $r > 1$,

$$\epsilon_r = \min\left\{\frac{1}{r}, \frac{1}{V_r}, \epsilon_{r-1}\right\}, \tag{3.12}$$

ϵ_1 being chosen arbitrarily ($\epsilon_1 > 0$). We then complete the definition of $\beta(t)$ by taking $\beta(t) = \alpha(p_1)$ for $t > p_1$ and $\beta(t) = t\alpha(c)/c$ for $t \leqslant c$ unless $c = 0$, when we take $\beta(t) = 0$ for $t \leqslant 0$.

Since $\lim_{r\to\infty} \epsilon_r = 0$, by (3.12), $\beta(t)$ is continuous at the point c. Since $\beta(0) = 0$, while $\beta(t)$ is absolutely continuous in any closed interval

* If V_r is zero, we define $\epsilon_r = \min\{1/r, \epsilon_{r-1}\}$.

which does not contain c, it follows that $\beta(t) \in (CI)$. It therefore only remains to obtain (3.5).

It follows without difficulty from (3.7), (3.9), and (3.12) that

$$\left|\int\limits_{c}^{p_{r+1}} \beta(t)\,d\sigma_{n_r}(t)\right| < 2M||\alpha||+1+\epsilon_{r+1}V_{r+1}$$
$$\leqslant 2M||\alpha||+2.$$

Further, if we integrate by parts and use (3.6) and (3.8), we find that

$$\left|\int\limits_{p_r}^{1} \beta(t)\,d\sigma_{n_r}(t)\right| \leqslant 2M||\alpha||+\left|\int\limits_{p_r}^{1} \beta'(t)\sigma_{n_r}(t)\,dt\right|$$
$$\leqslant 2M||\alpha||+MA_r,$$

and, in the case $c > 0$,

$$\left|\int\limits_{0}^{c} \beta(t)\,d\sigma_{n_r}(t)\right| = \left|\frac{\alpha(c)}{c}\left\{c\sigma_{n_r}(c)-\int\limits_{0}^{c} \sigma_{n_r}(t)\,dt\right\}\right| \leqslant 2M||\alpha||.$$

In view of (3.10) and (3.11), it follows that

$$\int\limits_{0}^{1} \beta(t)\,d\sigma_{n_r}(t) > r-2.$$

Hence (3.5) is satisfied and the theorem established.

THEOREM 3. *In order that*

$$\overline{\lim_{n\to\infty}}\,|F_n(y)| < \infty \tag{3.13}$$

whenever $y(t) \in (CI)$, it is necessary and sufficient that

$$\overline{\lim_{n\to\infty}}\,||F_n|| < \infty. \tag{3.14}$$

Since (3.14) is a necessary and sufficient condition* for (3.13) to hold for all $y(t)$ of the B-space (C_0), Theorem 3 follows from Theorem 2.

4. Linear functionals in the space (I)

Given any function $x(t)$ of (I), I shall write†

$$y(t) = \int\limits_{0}^{t} x(u)\,du \quad (0 \leqslant t \leqslant 1), \tag{4.1}$$

the integrals involved being taken in the appropriate sense. I shall denote by $f(x)$ a functional defined in (I) and shall write

$$F(y) = f(x)$$

for all $y(t) = \int\limits_{0}^{t} x(u)\,du$ of (CI).

* Cf. H. Hahn (**7**), 5–7 or S. Banach (**5**), 80.

† Cf. A. Alexiewicz (**2**), 290; the arguments given by Alexiewicz are repeated as they are needed for the subsequent work.

Since $||y|| = ||x||$, it is easily verified that (4.1) gives a linear and isometric transformation of the space (I) into the space (CI). It follows that $f(x)$ is linear in (I) if and only if $F(y)$ is linear in (CI) and that, in this case,

$$||f|| = ||F||,$$

$||f||$, $||F||$ denoting norms in (I) and (CI) respectively.

It has already been pointed out that the general linear functional in the space (CI) is given by an equation of the form

$$F(y) = \int_0^1 y(t)\, d\theta(t),$$

where $\theta(t) \in (BV)$. By subtracting a constant from $\theta(t)$, if necessary, I shall suppose that $\theta(1) = 0$; it then follows from Theorem 1 that

$$||F|| = V_e(\theta) + |\theta(1-0)|.$$

Since

$$\int_0^1 y(t)\, d\theta(t) = -\int_0^1 x(t)\theta(t)\, dt = \int_0^1 x(t)k(t)\, dt,$$

whenever $y(t) = \int_0^t x(u)\, du$, $\theta(t) \in (BV)$, $\theta(1) = 0$, and $k(t) \cong -\theta(t)$, we can therefore state the following theorem:*

Theorem 4. *In order that $f(x)$ be linear in (I), it is necessary and sufficient that $f(x)$ be given by an equation of the form*

$$f(x) = \int_0^1 x(t)k(t)\, dt,$$

where $k(t) \in (EBV)$; moreover, $||f||$ is then given by the equation

$$||f|| = V_e(k) + \operatorname*{ess\text{-}lim}_{t \to 1-0} |k(t)|.$$

We now obtain theorems concerning a sequence of linear functionals. Throughout the rest of the paper $f_n(x)$ will denote a functional linear in (I).

Theorem 5. *In order that*†

$$\overline{\lim_{n\to\infty}}\, |f_n(x)| < \infty$$

whenever $x(t) \in (I)$, it is necessary and sufficient that

$$\overline{\lim_{n\to\infty}}\, ||f_n|| < \infty. \tag{4.2}$$

* Cf. A. Alexiewicz, loc. cit.

† If (I) were complete (or, more generally, of the second category in itself) Theorem 5 would follow from a known result of S. Banach and H. Steinhaus [(**3**), 53]. It should be observed, however, that the space (I), like the space (CI), is of the first category in itself.

Theorem 5 follows from Theorem 3 which gives the corresponding result for the space (CI).

In view of Theorem 5, it follows by standard arguments that if

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all $x(t)$ of (I), where $f(x)$ is finite, then $f(x)$ is additive and bounded,* and therefore linear, in (I). Further, we can state† the following theorem:

THEOREM 6. *In order that*

$$\lim_{n\to\infty} f_n(x) = f(x) \tag{4.3}$$

for all $x(t)$ of (I), where $f(x)$ is linear in (I), it is necessary and sufficient that

$$\overline{\lim_{n\to\infty}}\, ||f_n|| < \infty, \tag{4.2}$$

and that (4.3) *should hold in a fundamental sub-set of* (I).

I now write, for $n = 1, 2, 3,\ldots$,

$$f_n(x) = \int_0^1 x(t)k_n(t)\,dt,$$

where $k_n(t) \in (EBV)$, define

$$M_e(k_n) = \underset{0<t<1}{\text{ess-}\overline{\text{bound}}}\, |k_n(t)|,$$

and take $\rho_n(t)$ such that $\rho_n(t) \in (BVN)$ and $\rho_n(t) \simeq k_n(t)$. Then it follows from Theorem 4 and the definition of $\rho_n(t)$ that

$$||f_n|| = V_e(k_n)+|\rho_n(1)| = \int_0^1 |d\rho_n(t)|+|\rho_n(1)|. \tag{4.4}$$

Since, for $0 \leqslant t \leqslant 1$,

$$|\rho_n(t)-\rho_n(1)| \leqslant \int_0^1 |d\rho_n(t)|,$$

it therefore follows that

$$M_e(k_n) \leqslant ||f_n||. \tag{4.5}$$

I next obtain a lemma, and then interpret Theorems 5 and 6 in terms of the sequence $\left\{\int_0^1 x(t)k_n(t)\,dt\right\}$.

* i.e. $|f(x)| \leqslant K||x||$ for all $x(t)$ of (I), where K is constant.

† Cf. S. Banach and H. Steinhaus (3), 53–4 and S. Banach (5), 123.

LEMMA. *If*
$$\overline{\lim_{n\to\infty}}\left|\int_0^1 k_n(t)\,dt\right| < \infty, \tag{4.6}$$
then, in order that
$$\overline{\lim_{n\to\infty}}\, ||f_n|| < \infty, \tag{4.2}$$
it is necessary and sufficient that
$$\overline{\lim_{n\to\infty}}\, V_e(k_n) < \infty. \tag{4.7}$$

In view of (4.4), it will be sufficient to show that (4.6) and (4.7) imply that*
$$\overline{\lim_{n\to\infty}}\, |\rho_n(1)| < \infty. \tag{4.8}$$

Suppose, if possible, that (4.6) and (4.7) are satisfied, but that (4.8) does not hold. Take K such that
$$V_e(k_n) \leqslant K \quad (n = 1, 2, 3,...),$$
and let $\{n_r\}$ be an increasing sequence of positive integers such that
$$|\rho_{n_r}(1)| > K+r \quad (r = 1, 2, 3,...).$$
Since, for $0 \leqslant t \leqslant 1$ and $n = 1, 2, 3,...,$
$$|\rho_n(t)-\rho_n(1)| \leqslant \int_0^1 |d\rho_n(t)| = V_e(k_n) \leqslant K,$$
it follows that, for every positive integer r,
$$|\rho_{n_r}(t)| > r \quad (0 \leqslant t \leqslant 1),$$
$\rho_{n_r}(t)$ being of constant sign for $0 \leqslant t \leqslant 1$. Since this is clearly impossible if (4.6) is satisfied, (4.8) must hold and the lemma is established.

THEOREM 7. *If $k_n(t) \in (EBV)$ for $n = 1, 2, 3,...,$ then in order that*
$$\overline{\lim_{n\to\infty}}\left|\int_0^1 x(t)k_n(t)\,dt\right| < \infty$$
whenever $x(t) \in (I)$, it is necessary and sufficient that
$$\overline{\lim_{n\to\infty}}\, V_e(k_n) < \infty \tag{4.7}$$
and
$$\overline{\lim_{n\to\infty}}\, M_e(k_n) < \infty. \tag{4.9}$$

It follows from (4.4), (4.5), and the definition of $\rho_n(t)$ that
$$\max\{M_e(k_n), V_e(k_n)\} \leqslant ||f_n|| \leqslant V_e(k_n)+M_e(k_n).$$
Thus (4.7) and (4.9) are equivalent to (4.2), and Theorem 7 follows from Theorem 5.

* For a similar result see J. C. Burkill (**6**), 129.

THEOREM 8. *If $k(t) \in (EBV)$, and $k_n(t) \in (EBV)$ for $n = 1, 2, 3,\ldots$, then in order that*

$$\lim_{n\to\infty} \int_0^1 x(t)k_n(t)\,dt = \int_0^1 x(t)k(t)\,dt \tag{4.10}$$

whenever $x(t) \in (I)$, it is necessary and sufficient that*

$$\overline{\lim_{n\to\infty}}\, V_e(k_n) < \infty \tag{4.7}$$

and

$$\lim_{n\to\infty} \int_0^\lambda k_n(t)\,dt = \int_0^\lambda k(t)\,dt \quad (0 < \lambda \leqslant 1). \tag{4.11}$$

Since the set of step-functions (defined for $0 \leqslant t \leqslant 1$) is dense† in (I), a fundamental sub-set of (I) consists of the characteristic functions of intervals $[0, \lambda]$, where λ takes all values such that $0 < \lambda \leqslant 1$. It therefore follows from Theorem 6 that (4.2) and (4.11) are necessary and sufficient for (4.10) to hold whenever $x(t) \in (I)$. The result stated therefore follows from the lemma.

* For a set of sufficient conditions in the case of the special Denjoy integral, see J. C. Burkill (**6**), 129.

† Cf. J. C. Burkill (**6**), 128.

REFERENCES

1. C. R. Adams and A. P. Morse, 'On the space (BV)', *Trans. American Math. Soc.* 42 (1937), 194–205.
2. A. Alexiewicz, 'Linear functionals on Denjoy-integrable functions', *Colloquium Math.* 1 (1948), 289–93.
3. S. Banach and H. Steinhaus, 'Sur le principe de la condensation de singularités', *Fundamenta Math.* 9 (1927), 50–61.
4. S. Banach, 'Über die Bairesche Kategorie gewisser Funktionenmengen', *Studia Math.* 3 (1931), 174–9.
5. S. Banach, *Théorie des opérations linéaires* (Warsaw, 1932).
6. J. C. Burkill, 'On Hobson's convergence theorem for Denjoy integrals', *J. of London Math. Soc.* 4 (1929), 127–32.
7. H. Hahn, 'Über Folgen linearer Operationen', *Monats. für Math.* 32 (1922), 3–88.
8. H. Lebesgue, 'Sur les intégrales singulières', *Annales de Toulouse* (3), 1 (1909), 25–117.
9. F. Riesz, 'Sur les opérations fonctionnelles linéaires', *Comptes Rendus*, 149 (1909), 974–7.
10. W. L. C. Sargent, 'On the integrability of a product', *J. of London Math. Soc.* 23 (1948), 28–34.
11. D. V. Widder, *The Laplace Transform* (Princeton, 1946).

CALCULATION OF THE HOMOTOPY GROUPS OF A_n^2-POLYHEDRA (I)

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[Received 24 October 1949]

1. Introduction

A (finite) connected polyhedron K of at most $n+2$ dimensions has been described by J. H. C. Whitehead as an A_n^2-polyhedron if its first $n-1$ homotopy groups† vanish. In (1) he showed that, if $n > 2$, the homotopy type of such a polyhedron is determined by its rth absolute cohomology groups with integer coefficients ($r = n, n+1, n+2$); its sth cohomology groups (mod 2) ($s = n, n+2$); and three related homomorphisms.

The object of this paper is to obtain an explicit expression for $\pi_{n+1}(K)$ in terms of the cohomology groups of K, together with the related homomorphisms. Of course, the nth homotopy group of an A_n^2-polyhedron is isomorphic to the nth absolute homology group with integer coefficients,‡ the isomorphism being induced by the natural mapping of homotopy into homology groups. In a later paper, the groups $\pi_{n+2}(K)$ and $\pi_{n+3}(K)$ will be considered. These groups will involve unknown homotopy groups of spheres, and, in fact, the fundamental difficulty in extending the methods of this paper to the calculation of the higher homotopy groups is to compute the homotopy groups of spheres.

Actually, we consider the homology system of K rather than its cohomology system. This dual system consists of $H_n(K)$, $H_{n+1}(K)$, $H_{n+2}(K)$, the absolute homology groups with integer coefficients; $H_n(K, 2)$, $H_{n+2}(K, 2)$, the homology groups (mod 2); and three homomorphisms. We obtain expressions for the homotopy groups in terms of homology, and, although the results may be reinterpreted in cohomology language, the formulations do not appear to be so natural.

2. A_n^2-cohomology and homology systems

I recall from (1) the definition of an A_n^2-cohomology system. It consists in the first place of three abelian groups labelled H^n, H^{n+1}, H^{n+2}. Each has a finite number of generators and H^n is free abelian. The

† For definition of the homotopy groups of a topological space, see (6) or (4), p. 410.

‡ See (6), 38, p. 521 et seq.

group $H^r(2)$ is a trivial extension of H^r_2 by† ${}_2H^{r+1}$, i.e. there exists an isomorphism $\Delta^*\colon {}_2H^{r+1} \to H^r(2)$ and a homomorphism $\Delta\colon H^r(2) \to H^{r+1}$, such that $\Delta\Delta^* = 1$, $\Delta^{-1}(0) = H^r_2$, and

$$H^r(2) = H^r_2 + \Delta^*({}_2H^{r+1}). \tag{2.1}$$

Δ being supposed given, Δ^* is not uniquely determined: in fact, it is determined *modulo* an arbitrary homomorphism ${}_2H^{r+1} \to H^r_2$; but we simply postulate that it possesses the required properties; μ is the natural homomorphism $H^r \to H^r(2)$, which maps H^r on to $\mu H^r = H^r_2$. Finally, a homomorphism

$$\gamma^*\colon H^n(2) \to H^{n+2}(2) \tag{2.2}$$

is given.

Such a system is provided by the cohomology groups of an A^2_n-polyhedron K. H^n, H^{n+1}, H^{n+2} then become the absolute cohomology groups $H^n(K)$, $H^{n+1}(K)$, $H^{n+2}(K)$, with integer coefficients. $H^n(2)$, $H^{n+2}(2)$ become the cohomology groups‡ (mod 2) $H^n(K, 2)$ and $H^{n+2}(K, 2)$.

Let x' be a co-cycle (mod 2) in the class $x \in H^r(K, 2)$. Then Δx is uniquely determined as the absolute cohomology class of $\frac{1}{2}\delta x'$. Let $H^{r+1}(K)$ be expressed as the direct sum of a free abelian group and (a finite number of) cyclic groups of finite order. Let y generate a cyclic summand of even order σ, let y' be a co-cycle in the class y, and let the r-dimensional co-chain v' be so chosen that $\delta v' = \sigma y'$. Then $\Delta^*\frac{1}{2}\sigma y$ may be defined as the cohomology class (mod 2) of v', and Δ^* may be extended over ${}_2H^{r+1}(K)$ by linearity to yield a homomorphism $\Delta^*\colon {}_2H^{r+1}(K) \to H^r(K, 2)$. It is obvious from the definitions that $\Delta\Delta^* = 1$. Replacement of v' by $v'' = v' + z$, where z is a co-cycle, changes $\Delta^*\frac{1}{2}\sigma y$ by the addition of an element in $H^r_2(K)$, so that Δ^* is determined *modulo* an arbitrary homomorphism ${}_2H^{r+1}(K) \to H^r_2(K)$. Further, μ is the homomorphism $H^r(K) \to H^r(K, 2)$, which is induced by regarding an absolute co-cycle as a co-cycle (mod 2), and its image is $H^r_2(K)$. Now let $x \in \Delta^{-1}(0)$, and let x' again be a co-cycle (mod 2) in the class x. Then $\frac{1}{2}\delta x' = \delta y'$, whence $x' - 2y'$ is a co-cycle. If v is its cohomology class, $\mu v = x$, so that $\Delta^{-1}(0) \subset \mu H^r(K)$. Since obviously $\Delta H^r_2(K) = 0$, it follows that $\Delta^{-1}(0) = \mu H^r(K) = H^r_2(K)$.

† If G is an abelian group, mG is the group of elements mg, $g \in G$; G_m is the difference group $G - mG$; and ${}_mG$ is the group of elements g such that $mg = 0$. In the definitions of Δ, Δ^* given here H^{n+3} is to be taken as zero, so that $H^{n+2}(2) = H^{n+2}_2$.

‡ x' is an r-dimensional co-cycle (mod 2) if it is an r-dimensional co-chain and $\delta x' \equiv 0 \pmod 2$; x' is cohomologous to x'' (mod 2) if there exists an $(r-1)$-dimensional co-chain y such that $x' - x'' \equiv \delta y \pmod 2$. See (2), 50.

Further, since $\Delta\Delta^* = 1$,

$$H^r(K,2) = \Delta^{-1}(0) + \Delta^*{}_2H^{r+1}(K)$$
$$= H_2^r(K) + \Delta^*{}_2H^{r+1}(K). \tag{2.3}$$

Finally, γ^* is the Steenrod square,†

$$\gamma^* x = x \cup_{n-2} x, \tag{2.4}$$

where $x \in H^n(K,2)$, $\gamma^* x \in H^{n+2}(K,2)$.

Whitehead has shown that any A_n^2-cohomology system is properly isomorphic‡ to the cohomology system of an A_n^2-polyhedron, and that, for $n > 2$, two A_n^2-polyhedra are of the same homotopy type if and only if their associated cohomology systems are properly isomorphic.

An A_n^2-homology system is defined similarly. In fact, algebraically, there is no distinction, and it is only in the interpretation of the abstract system in terms of the homology groups of an A_n^2-polyhedron and related homomorphisms that the distinction appears. It should be recalled that the cohomology and homology groups of a finite complex are connected by the relations

$$\left.\begin{aligned} B^r(K) &\approx B_r(K) \\ T^{r+1}(K) &\approx T_r(K) \end{aligned}\right\}, \tag{2.5}$$

where $B_r(K)$, $T_r(K)$ represent respectively the free abelian and the finite summand of $H_r(K)$, and $B^r(K)$, $T^r(K)$ are similarly defined. The abstract groups H^n, H^{n+1}, H^{n+2} of the system are now to be interpreted as $H_{n+2}(K)$, $H_{n+1}(K)$, $H_n(K)$; $H^n(2)$, $H^{n+2}(2)$ become $H_{n+2}(K,2)$, $H_n(K,2)$; and Δ, Δ^*, μ are translated into homology language by substituting the boundary for the co-boundary operator into their definitions. The homomorphism $\gamma^*\colon H^n(2) \to H^{n+2}(2)$ is interpreted in the homology description as a homomorphism

$$\gamma\colon H_{n+2}(K,2) \to H_n(K,2), \tag{2.6}$$

which is, in fact, dual to the Steenrod square [see (2.4)]. It is clear that such a system will also determine uniquely the homotopy type of K, for it determines uniquely and is uniquely determined by the cohomology system of K.

It is convenient to refer to the groups of an abstract A_n^2-homology system as H_{n+2}, H_{n+1}, H_n, $H_{n+2}(2)$, $H_n(2)$, and to refer to the homomorphisms

$$\Delta\colon H_{r+1}(2) \to H_r,\ \ \Delta^*\colon {}_2H_r \to H_{r+1}(2),\ \ \mu\colon H_r \to H_r(2),\ \ \gamma\colon H_{n+2}(2) \to H_n(2).$$

† For the definition of γ^*, see (7).

‡ Two systems are said by Whitehead to be 'properly isomorphic' if their individual group-constituents are isomorphic and the isomorphisms commute with Δ, μ, and γ^*.

Where no confusion can arise, it will also be convenient to use the symbols for the abstract system in referring to the corresponding system of a given polyhedron.

3. The reduced form and realization of γ

Whitehead has shown that any A_n^2-cohomology system can be realized by a reduced complex; that is to say, any A_n^2-complex is of the same homotopy type as a complex K of the following form:

$K^0 = K^{n-1} = e^0$, a single point;†

$K^n = e^0+e_1^n+...+e_m^n$, so that e^0 closes e_i^n to a sphere S_i^n;

$K^{n+1} = K^n+e_1^{n+1}+...+e_{t+l}^{n+1}$, where e_i^{n+1} $(1 \leqslant i \leqslant t \leqslant m)$ is attached to K^n by a map f_i: $\dot{E}_i^{n+1} \to S_i^n$ of degree σ_i (σ_i being odd if and only if $1 \leqslant i \leqslant h \leqslant t$), and e^0 closes e_{t+j}^{n+1} $(1 \leqslant j \leqslant l)$, to a sphere S_j^{n+1};

K^{n+2} is formed by attaching $(n+2)$-cells to $K^n+S_1^{n+1}+...+S_l^{n+1}$, so that e_i^{n+1} $(1 \leqslant i \leqslant t)$ are principal cells of the complex.

It is then easy to see that $H_n(K)$ is the direct sum of a free abelian group of rank $m-t$ and t cyclic summands of orders $\sigma_1,..., \sigma_t$. It is given by generators $a_1,..., a_m$, where a_i corresponds to the cell e_i^n, and relations $\sigma_i a_i = 0$ $(i = 1,..., t)$. $H_n(K, 2)$ is a free (mod 2) module freely generated by $\bar{a}_{h+1},..., \bar{a}_m$, where $\bar{a}_i$ is the residue class (mod 2) containing a_i.

I now construct the reduced complex realizing a given A_n^2-homology system. We may, of course, assume that H_n, $H_n(2)$ are isomorphic to the corresponding groups already obtained from K, and it remains to prescribe the precise form of the $(n+2)$-dimensional structure of K in order to realize H_{n+1}, H_{n+2}, $H_{n+2}(2)$ and the homomorphism γ.

Let $H_{n+1} = (b_1,..., b_l)$, where b_i is of order τ_i $(1 \leqslant i \leqslant u)$ (τ_i being odd if and only if $1 \leqslant i \leqslant k \leqslant u$), and b_{u+j} is of infinite order $(j = 1,..., l-u)$.

Let $H_{n+2} = (c_1,..., c_p)$, being free abelian. Then

$$\begin{aligned} H_{n+2}(2) &= \mu H_{n+2}+\Delta^*{}_2 H_{n+1} \\ &= (\bar{c}_1,..., \bar{c}_p)+(\Delta^*\tfrac{1}{2}\tau_{k+1} b_{k+1},..., \Delta^*\tfrac{1}{2}\tau_u b_u) \\ &= (\bar{c}_1,..., \bar{c}_p, \bar{c}_{p+1},..., \bar{c}_{p+u-k}). \end{aligned}$$

Let γ: $H_{n+2}(2) \to H_n(2)$ be given by

$$\gamma\bar{c}_i = \sum_{\gamma=h+1}^{m} \gamma_{ij}\bar{a}_j \quad (i = 1,..., p+u-k), \tag{3.1}$$

where, of course, the coefficients γ_{ij} can take only the values 0 or 1.

† If K is a complex, K^r is the sub-complex of K consisting of all cells of at most r dimensions. K^r is called the r-dimensional skeleton (or section) of K.

Consider now the K^{n+1} given in the definition of a reduced complex. Let $p+u$ $(n+2)$-cells be attached to K^{n+1} in the following way:

e_i^{n+2} $(i = 1,..., p)$ are attached by maps f_i: $\dot{E}_i^{n+2} \to \sum \gamma_{ij} S_j^n$;

e_{p+i}^{n+2} $(i = 1,..., u-k)$ are attached by maps f_{p+i}: $\dot{E}_{p+i}^{n+2} \to S_{k+i}^{n+1}+$
$+\sum \gamma_{p+i,j} S_j^n$; where f_{p+i} is of degree τ_{k+i} over S_{k+i}^{n+1};

$e_{p+u-k+i}^{n+2}$ $(i = 1,..., k)$ are attached by maps $f_{p+u-k+i}$: $\dot{E}_{p+u-k+i}^{n+2}$
$\to S_i^{n+1}$ of degree τ_i.

In these specifications $\gamma_{ij} S_j^n$ is to be understood as S_j^n if $\gamma_{ij} = 1$, and as e^0 if $\gamma_{ij} = 0$. The maps f_i $(i = 1,..., p+u-k)$ are to be understood as essential over those n-spheres S_j^n for which $\gamma_{ij} = 1$.

LEMMA 3.1. *The complex K, thus constructed, realizes the given A_n^2-homology system.*

For we have already seen that K realizes H_n and $H_n(2)$. The boundary homomorphism,† ∂: $C_{n+2}(K) \to C_{n+1}(K)$ is given by

$$\partial e_i^{n+2} = 0 \quad (i = 1,..., p),$$

$$\partial e_{p+i}^{n+2} = \tau_{k+i}\, e_{t+k+i}^{n+1} \quad (i = 1,..., u-k),$$

$$\partial e_{p+u-k+i}^{n+2} = \tau_i\, e_{t+i}^{n+1} \quad (i = 1,..., k).$$

$Z_{n+1}(K)$, the group of $(n+1)$-cycles, is the free abelian group freely generated by $e_{t+1}^{n+1},..., e_{t+l}^{n+1}$, and the group of $(n+1)$-boundaries is the sub-group freely generated by $\tau_1 e_{t+1}^{n+1},..., \tau_u e_{t+u}^{n+1}$. Thus $H_{n+1}(K)$ is generated by u elements of orders $\tau_1,..., \tau_u$ and $l-u$ elements of infinite order. This establishes the isomorphism of $H_{n+1}(K)$ with H_{n+1}.

$Z_{n+2}(K) = H_{n+2}(K)$ is freely generated by $e_1^{n+2},..., e_p^{n+2}$ and is therefore isomorphic to H_{n+2}. $H_{n+2}(K, 2)$ being obtained from $H_{n+2}(K)$ and $H_{n+1}(K)$ precisely as $H_{n+2}(2)$ is obtained from H_{n+2} and H_{n+1}, the isomorphism between $H_{n+2}(K, 2)$ and $H_{n+2}(2)$ is also established.

I recall from (1) the manner in which γ is defined in K. Since K is simply connected, we may identify $C_{n+2}(K)$ with $\pi_{n+2}(K, K^{n+1})$. The homotopy boundary operator β is then a homomorphism

$$\beta\colon C_{n+2}(K) \to \pi_{n+1}(K^{n+1}) \approx H_{n+1}(K^{n+1})+\pi_{n+1}(K^n)$$
$$\approx H_{n+1}(K^{n+1})+H_n(K, 2).$$

Identify‡ $\pi_{n+1}(K^{n+1})$ with $H_{n+1}(K^{n+1})+H_n(K, 2)$ and let $\beta c = c_1+c_2$,

† $C_r(K)$ stands as usual for the group of r-dimensional chains of K.

‡ For further clarification, see § 5 of (1) or § 4 of this paper.

$c \in C_{n+2}(K)$, $c_1 \in H_{n+1}(K^{n+1})$, $c_2 \in H_n(K, 2)$. Then $c \to c_2$ is a homomorphism

$$\theta\colon C_{n+2}(K) \to H_n(K, 2).$$

The homomorphism γ is obtained by defining

$$\gamma \bar{e}_i^{n+2} = \theta e_i^{n+2} \; (i = 1,..., p+u-k).$$

This definition is unique since every element in $H_n(K, 2)$ satisfies $2x = 0$. Now it follows from the way in which e_i^{n+2} is attached to K^{n+1} that

$$\theta e_i^{n+2} = \sum_{j=h+1}^{m} \gamma_{ij} \bar{a}_j.$$

Thus γ is given by

$$\gamma \bar{e}_i^{n+2} = \sum_{j=h+1}^{m} \gamma_{ij} \bar{a}_j \quad (i = 1,..., p+u-k),$$

and therefore realizes (3.1).

This establishes Lemma 3.1. The argument is implied, but not explicitly stated by Whitehead in (**1**), though he provided an explicit proof in (**2**) of the realizability of an A_2^2-cohomology system, arising in the (more difficult) case of a four-dimensional polyhedron.

The cohomology groups of K may be obtained using the relations (2.5). From these it follows that

$H^n(K)$ is free abelian of rank $m-t$;

$H^{n+1}(K)$ is the direct sum of a free abelian group of rank $l-u$ and t cyclic summands of orders $\sigma_1,..., \sigma_t$;

$H^{n+2}(K)$ is the direct sum of a free abelian group of rank p, and u cyclic summands of orders $\tau_1,..., \tau_u$.

$H^n(K, 2)$, $H^{n+2}(K, 2)$, being finite abelian groups, are isomorphic to $H_n(K, 2)$, $H_{n+2}(K, 2)$, the isomorphism depending on the choice of (mod 2) basis. If bases $\{\bar{a}^i\}$, $\{\bar{c}^j\}$ are chosen for $H^n(K, 2)$, $H^{n+2}(K, 2)$ dual to the bases $\{\bar{a}_i\}$, $\{\bar{c}_j\}$ for $H_n(K, 2)$, $H_{n+2}(K, 2)$, already given, then γ^* is given by

$$\gamma^* \bar{a}^i = \sum_{j=1}^{p+u-k} \gamma_{ji} \bar{c}^j \quad (i = h+1,..., m).$$

It is not difficult to see how, by reversing the above analysis, and thus translating cohomology requirements into homology requirements, a given cohomology system may be realized by a reduced complex.

4. $\pi_{n+1}(K)$

I start this section by quoting Whitehead's fundamental suspension theorem,† which plays a paramount role in obtaining the homotopy groups of a cell complex.

† See (**3**), Theorem 2.

LEMMA 4.1. (Partial statement of Whitehead's theorem.) *Let X be an arcwise-connected topological space and let a t-cell be attached by a map $f\colon \dot{E}^t \to X$. Let f induce the homomorphism $h_s\colon \pi_s(S^{t-1}) \to \pi_s(X)$, and let i_s be the homomorphism $\pi_s(X) \to \pi_s(X+e^t)$ induced by the identity map. Then, if $\pi_s(X) = 0$ $(1 \leqslant s \leqslant r-t+2;\ r < 2t-2)$,*

$$\pi_r(X+e^t) - i_r\,\pi_r(X) \approx h_{r-1}^{-1}(0), \tag{4.1}$$

$$i_r^{-1}(0) = h_r\,\pi_r(S^{t-1}). \tag{4.2}$$

By using the exactness of the homotopy sequence of the pair $(X+e^t, X)$, this result may be re-expressed (and somewhat generalized) as Theorem 1 of (**3**).

We now proceed to determine the $(n+1)$th homotopy group of the reduced complex K described in § 3. Since $n > 2$, it follows by repeated application of the result in § 8 of (**1**) that

$$\pi_{n+1}(K^n) \approx \pi_{n+1}(S_1^n) + \dots + \pi_{n+1}(S_m^n), \tag{4.3}$$

so that $\pi_{n+1}(K^n)$ is a free (mod 2) module on m generators. Repeated application of Lemma 4.1 shows that the attachment of an $(n+1)$-cell bounded by an n-cell taken an odd number of times annihilates the corresponding generator of $\pi_{n+1}(K^n)$, the attachment of an $(n+1)$-cell bounded by an n-cell taken an even number of times has no effect, and the attachment of an $(n+1)$-sphere adds a cyclic infinite direct summand. Combining these results we have

$$\pi_{n+1}(K^{n+1}) \approx H_n(K,2) + H_{n+1}(K^{n+1}). \tag{4.4}$$

It is now necessary to consider the effect of adding $(n+2)$-cells. Since $\pi_n(S^{n+1}) = 0$, it follows that (4.1) reduces to the simpler form

$$\pi_{n+1}(X+e^{n+2}) = i_{n+1}\,\pi_{n+1}(X). \tag{4.5}$$

In other words, $\pi_{n+1}(X+e^{n+2})$ is obtained from $\pi_{n+1}(X)$ by adding the relation $i_{n+1}^{-1}(0) = 0$, or, using (4.2), $h_{n+1}\,\pi_{n+1}(S^{n+1}) = 0$. Thus $\pi_{n+1}(K)$ is obtained from $\pi_{n+1}(K^{n+1})$ by adding the $p+u$ relations corresponding to the $p+u$ $(n+2)$-cells which are attached.

I write $H_{n+1}(K^{n+1}) = (b_1^*, \dots, b_l^*)$, $H_n(K,2) = (\bar{a}_{h+1}, \dots, \bar{a}_m)$. We identify $\pi_{n+1}(K^{n+1})$ with $H_n(K,2) + H_{n+1}(K^{n+1})$, so that

$$\pi_{n+1}(K^{n+1}) = (\bar{a}_{h+1}, \dots, \bar{a}_m) + (b_1^*, \dots, b_l^*). \tag{4.6}$$

Consider first the relation introduced by the attachment of a typical one of the first p $(n+2)$-cells. Let the attaching map be

$$f\colon \dot{E}^{n+2} \to S_\alpha^n + \dots + S_\beta^n.$$

Then $h_{n+1}\,\pi_{n+1}(\dot{E}^{n+2}) = (\bar{a}_\alpha + \dots + \bar{a}_\beta)$, in view of the above identification,

and the relation introduced is $\bar{a}_\alpha+...+\bar{a}_\beta = 0$. Thus each e_i^{n+2} $(i = 1,..., p)$ contributes the relation $\sum \gamma_{ij}\bar{a}_j = 0$,

i.e. $$\gamma\bar{c}_i = 0 \quad (i = 1,..., p). \tag{4.7}$$

Now consider the relation introduced by the attachment of a typical one of the next $u-k$ $(n+2)$-cells. Let the attaching map be

$$f: \dot{E}^{n+2} \to S^{n+1}+S^n_\lambda+...+S^n_\mu,$$

where f is of even degree τ over S^{n+1} and essential over each of the n-spheres $S^n_\lambda,..., S^n_\mu$. Then $h_{n+1}\pi_{n+1}(\dot{E}^{n+2})$ is the cyclic group generated by $\tau b^*+\bar{a}_\lambda+...+\bar{a}_\mu$, where b^* generates $\pi_{n+1}(S^{n+1})$ and is therefore a generator of $H_{n+1}(K^{n+1})$. Thus each e^{n+2}_{p+i} $(i = 1,..., u-k)$ contributes the relation

$$\tau_{k+i}b^*_{k+i}+\sum \gamma_{p+i,j}\bar{a}_j = 0,$$

i.e. $$\tau_{k+i}b^*_{k+i}+\gamma\bar{c}_{p+i} = 0 \quad (i = 1,..., u-k). \tag{4.8}$$

A similar, but simpler, argument shows that the effect of attaching the $(n+2)$-cells $e^{n+2}_{p+u-k+i}$ $(i = 1,..., k)$ is to add the relations

$$\tau_i b^*_i = 0 \quad (i = 1,..., k). \tag{4.9}$$

Combining (4.7), (4.8), and (4.9), we have the result that $\pi_{n+1}(K)$ is obtained from $H_n(K,2)+H_{n+1}(K^{n+1}) = H_n(K,2)+(b^*_1,..., b^*_l)$ by adding the relations

$$\left.\begin{aligned} \gamma\bar{c}_i &= 0 \quad (i = 1,..., p) \\ \tau_{k+i}b^*_{k+i}+\gamma\bar{c}_{p+i} &= 0 \quad (i = 1,..., u-k) \\ \tau_i b^*_i &= 0 \quad (i = 1,..., k) \end{aligned}\right\}. \tag{4.10}$$

This formulation of the result is not satisfactory because $H_{n+1}(K^{n+1})$ is not a group of the homology system, not being, of course, an invariant of homotopy type.

Let the (abstract) group H_{n+1} be written as $B_{n+1}+T^1_{n+1}+T^2_{n+1}$, where B_{n+1} is its free abelian direct summand, T^1_{n+1} is generated by $b_1,..., b_k$, of odd orders, and T^2_{n+1} is generated by $b_{k+1},..., b_u$, of even orders. Replace T^2_{n+1} by a free abelian group freely generated by $b^*_{k+1},..., b^*_u$, and let the group resulting from H_{n+1} by this replacement be called H^*_{n+1}. Then it follows from (4.10) that $\pi_{n+1}(K)$ is obtained from $H_n(2)+H^*_{n+1}$ by adding the relations

$$\gamma\bar{c}_i = 0 \quad (i = 1,..., p),$$

$$\gamma\bar{c}_{p+1} = \tau_{k+i}b^*_{k+i} \quad (i = 1,..., u-k).$$

Let i be the natural homomorphism $H_n(2) \to H_n(2)-\gamma\mu H_{n+2}$ and let γ_1: $\Delta^*_2 H_{n+1} \to H_n(2)-\gamma\mu H_{n+2}$ be the homomorphism

$$i\gamma: H_{n+2}(2) \to H_n(2)-\gamma\mu H_{n+2}$$

restricted to $\Delta_2^* H_{n+1}$. Then it follows that $\pi_{n+1}(K)$ is obtained from $H_{n+1}^* + (H_n(2) - \gamma\mu H_{n+2})$ by adding the relations

$$\gamma_1 \bar{c}_{p+i} = \tau_{k+i} b_{k+i}^* \quad (i = 1,..., u-k). \tag{4.11}$$

This formulation shows that $\pi_{n+1}(K)$ is a group extension† of H_{n+1} by $H_n(2) - \gamma\mu H_{n+2}$; that is to say, a homomorphism ϕ of $\pi_{n+1}(K)$ onto H_{n+1} is given whose kernel is $H_n(2) - \gamma\mu H_{n+2}$. The homomorphism is, of course, the natural homomorphism of $\pi_{n+1}(K)$ into $H_{n+1}(K)$, which is onto because K is aspherical in the first $(n-1)$-dimensions. The relations (4.11) are obviously consistent with ϕ because

$$\phi\gamma_1 \bar{c}_{p+i} \in \phi(H_n(2) - \gamma\mu H_{n+2}) = 0,$$

and $\phi\tau_{k+i} b_{k+i}^* = \tau_{k+i} b_{k+i} = 0$.

We may write $\pi_{n+1}(K) \approx B_{n+1} + T_{n+1}^1 + E$, where E is a group extension of T_{n+1}^2 by $H_n(2) - \gamma\mu H_{n+2}$. The generators $b_{k+1}^*,..., b_u^*$ of E 'generate' a particular choice of representatives of the elements of T_{n+1}^2 in E; namely, we choose as the representative of the element

$$\sum_{i=1}^{u-k} \lambda_{k+i} b_{k+i} \quad (0 \leqslant \lambda_{k+i} < \tau_i)$$

the element $\sum_{i=1}^{u-k} \lambda_{k+i} b_{k+i}^*$ in E. The factor sets corresponding to this choice of representatives—and therefore the group extension itself—are fully determined when the elements $\tau_{k+i} b_{k+i}^* \in H_n(2) - \gamma\mu H_{n+2}$ are given.‡ It follows from (4.11) that $\tau_{k+i} b_{k+i}^* = i\gamma\Delta^{-1} \frac{1}{2}\tau_{k+i} b_{k+i}$.

THEOREM 4.1. *If K is an A_n^2-polyhedron $(n > 2)$, then $\pi_{n+1}(K)$ is a group extension of H_{n+1} by*

$$H_n(2) - \gamma\mu H_{n+2}.$$

If H_{n+1} is given as $B_{n+1} + T_{n+1}^1 + T_{n+1}^2$, where B_{n+1} is the free abelian direct summand, T_{n+1}^1 is generated by elements of odd order and T_{n+1}^2 is generated by elements of even order, then $\pi_{n+1}(K) \approx B_{n+1} + T_{n+1}^1 + E$, where E is an extension of T_{n+1}^2 by $H_n(2) - \gamma\mu H_{n+2}$. Let T_{n+1}^2 be generated by $b_{k+1},..., b_u$ of orders $\tau_{k+1},..., \tau_u$. Then representatives of $b_{k+1},..., b_u$ may be chosen in E, and called $b_{k+1}^*,..., b_u^*$, such that

$$\tau_{k+i} b_{k+i}^* = i\gamma\Delta^{-1} \tfrac{1}{2}\tau_{k+i} b_{k+i}, \tag{4.12}$$

† In (5), J. H. C. Whitehead has an argument whose conclusion simplifies to a statement of π_{n+1} as a group extension when his case $(n = 2)$ is replaced by our case $(n > 2)$. Cf. also the relation $H_{n+1} \approx \pi_{n+1} - \eta\pi_n$ in (8), where η is the homomorphism $\pi_n(X) \to \pi_{n+1}(X)$ induced by an essential map of S^{n+1} on to S^n $(n > 2)$.

‡ In the Eilenberg–Maclane–Eckmann cohomology theory of abstract groups, $\pi_{n+1}(K)$, as an extension of H_{n+1} by $H_n(2) - \gamma\mu H_{n+2}$, is determined up to equivalence by an element of $H^2(H_{n+1}, H_n(2) - \gamma\mu H_{n+2})$.

where i is the natural homomorphism $H_n(2) \to H_n(2) - \gamma\mu H_{n+2}$. The relations (4.12) *determine the group extension.*

A special case which gives rise to a particularly simple formulation may be described algebraically by the condition $\gamma\Delta^*{}_2 H_{n+1} = 0$. Given this condition, γ_1 is the trivial homomorphism which maps every element of $\Delta^*{}_2 H_{n+1}$ to the zero element of $H_n(2) - \gamma\mu H_{n+2}$. Thus the relations (4.11) reduce to the simple relations $\tau_{k+i} b^*_{k+i} = 0$ and these relations have the effect of converting H^*_{n+1} back into H_{n+1}. Since in this special case $\gamma\mu H_{n+2} = \gamma H_{n+2}(2)$, we have† the

COROLLARY 4.1. *If γ is zero on $\Delta^*{}_2 H_{n+1}$,*

$$\pi_{n+1}(K) \approx H_{n+1} + \{H_n(2) - \gamma H_{n+2}(2)\}. \tag{4.13}$$

Geometrically, this case arises when, for example, there is either no $(n+1)$-dimensional torsion or only odd $(n+1)$-dimensional torsion.

5. $\pi_{n+1}(K)$ in terms of cohomology

The translation of the results of § 4 into the language of cohomology is achieved with the help of the relations (2.5), namely $B_r(K) \approx B^r(K)$, $T_r(K) \approx T^{r+1}(K)$.

It follows that the group H^*_{n+1} is isomorphic to $B^{n+1}(K) + T^{*n+2}(K)$, where $T^{*n+2}(K)$ is obtained from $T^{n+2}(K)$ by substituting generators of infinite order $b^{*k+1}, \ldots, b^{*u}$ for the generators $b^{k+1}, \ldots, b^u$, which are dual to $b_{k+1}, \ldots, b_u$. Also the group $H_n(2) - \gamma\mu H_{n+2}$, being a finite abelian group, is isomorphic to its character group. I prove that its character group is $\gamma^{*-1}(\mu T^{n+2})$. For the character group of $H_n(2) - \gamma\mu H_{n+2}$ consists of those elements in $H^n(2)$ which annihilate $\gamma\mu H_{n+2}$. Since, for $v \in H^n(2)$, $x \in H_{n+2}(2)$, $v(\gamma x) = (\gamma^* v)x$, it follows that

$$v \in Ch\{H_n(2) - \gamma\mu H_{n+2}\}$$

if and only if $\gamma^* v$ is zero on μH_{n+2}. It then follows from the choice of dual bases that $\gamma^* v \in \mu T^{n+2}$, $v \in \gamma^{*-1}(\mu T^{n+2})$.

I recall the homomorphism $\gamma_1 \colon \Delta^*{}_2 H_{n+1} \to H_n(2) - \gamma\mu H_{n+2}$ of § 4. Its dual, γ_1^*, is a homomorphism

$$\gamma_1^* \colon \gamma^{*-1}(\mu T^{n+2}) \to \mu T^{n+2}, \tag{5.1}$$

given by $\gamma_1^* y = \gamma^* y$, $y \in \gamma^{*-1}(\mu T^{n+2})$.

Choose any basis (mod 2) for $\gamma^{*-1}(\mu T^{n+2})$, say $(\bar{d}^1, \ldots, \bar{d}^\rho)$, and the

† We might just as well have argued that, in this case, E is the trivial extension of T^2_{n+1} by $H_n(2) - \gamma\mu H_{n+2}$.

particular basis (mod 2) for μT^{n+2} dual to $(\bar{c}_{p+1},..., \bar{c}_{p+u-k})$, and let γ_1^* be given by

$$\gamma_1^* \bar{d}^i = \sum_{j=1}^{u-k} \lambda_{ij} \bar{c}_{p+j} \quad (i = 1,..., \rho). \tag{5.2}$$

Then reference to Theorem 4.1 enables us to calculate $\pi_{n+1}(K)$ in terms of the cohomology system.

THEOREM 5.1. *With the above notation and $n > 2$, $\pi_{n+1}(K)$ is obtained from $B^{n+1}+T^{*n+2}+\gamma^{*-1}(\mu T^{n+2})$ by adding the relations*

$$\tau_{k+i} b^{*k+i} = \sum_{j=1}^{\rho} \lambda_{ji} d^j. \tag{5.3}$$

Now the character group of $H_n(2)-\gamma H_{n+2}(2)$ is simply $\gamma^{*-1}(0)$. Also the condition $\gamma\Delta^*{}_2 H_{n+1} = 0$ is equivalent to the condition

$$\gamma^* H^n(2) \subset \mu B^{n+2}.$$

This enables us to translate Corollary 4.1 into

COROLLARY 5.1. *If $\gamma^* H^n(2) \subset \mu B^{n+2}$, then*

$$\pi_{n+1}(K) \approx B^{n+1}+T^{n+2}+\gamma^{*-1}(0). \tag{5.4}$$

Perhaps the neatest statement of the result in this case combines Corollaries 4.1 and 5.1, giving

COROLLARY 5.2. *If $\gamma^* H^n(2) \subset \mu B^{n+2}$, then*

$$\pi_{n+1}(K) \approx H_{n+1}+\gamma^{*-1}(0). \tag{5.5}$$

The author wishes to thank Professor J. H. C. Whitehead for his very kind help in the preparation of this paper.

REFERENCES

1. J. H. C. Whitehead, 'The homotopy type of a special kind of polyhedron', *Annales Soc. Polonaise* (1948), 176–86.
2. —— 'On simply-connected 4-dimensional polyhedra', *Comm. Math. Helvetici* 22 (1949), 48–92.
3. —— 'A note on suspension', *Quart. J. of Math.* (Oxford), (2) *supra*, 9–22.
4. —— 'On adding relations to homotopy groups', *Annals of Math.* 42 (1941), 409–28.
5. —— 'A certain exact sequence', ibid. 51 (1950) (in the press).
6. W. Hurewicz, 'Topologie der Deformationen', *Kon. Wetensch. Amsterdam*, 38 (1935), 112–19, 521–8; 39 (1936), 117–25, 215–24.
7. N. E. Steenrod, 'Products of co-cycles and extensions of mappings', *Annals of Math.* 48 (1947), 290–320.
8. G. W. Whitehead, 'On spaces with vanishing low-dimensional homotopy groups', *Proc. National Acad. Sci.* 34 (1948), 207–11.

SOME SUSPENSION THEOREMS

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[Received 19 November 1949; in revised form 26 March 1950]

1. Statement of Theorems

LET $Y = X \cup \{e_\lambda^n\}$ $(n > 1)$ be an arcwise-connected Hausdorff space, which consists of a closed sub-space X and a set $\{e_\lambda^n\}$ of n-cells, each of which is an open sub-set of Y. The closure $\bar{e}_\lambda^n$ of each e_λ^n is to be the image of an n-simplex σ^n in a map ϕ_λ such that $\phi_\lambda \dot{\sigma}^n \subset X$ and $\phi_\lambda | \sigma^n - \dot{\sigma}^n$ is a homeomorphism onto e_λ^n, where $\dot{\sigma}^n$ is the boundary of σ^n. The set $\{e_\lambda^n\}$ may be infinite provided that the topology of Y is such that any compact sub-set of Y is contained in the union of X and a finite sub-set of the cells e_λ^n [cf. (**1**)]. We assume that, if $\pi_1(Y) \neq 1$, then Y is locally simply-connected in the weak sense, which permits the construction of a universal covering space $\tilde{Y}$.

Let $\xi_\lambda \in \pi_{n-1}(X)$ be the element which is represented by $\phi_\lambda | \dot{\sigma}^n$ and let

$$h_\lambda : (n-1)^r \to \pi_r(X) \quad (r > 0)$$

be the homomorphism of $(n-1)^r = \pi_r(\dot{\sigma}^n)$ which is induced by $\phi_\lambda | \dot{\sigma}^n$, with suitable conventions concerning base points. Let $i_r : \pi_r(X) \to \pi_r(Y)$ be the injection. By an *invariant sub-group* of $\pi_r(X)$ $(r \geqslant 1)$ we mean one which is invariant under the operators in $\pi_1(X)$. The following two theorems generalize certain results in (**1**) and (**2**).

THEOREM 1. *Let $r < 2n-2$ and let $\pi_s(X) = 0$ if $2 \leqslant s \leqslant r-n+1$. Then $i_r^{-1}(0)$ is the smallest invariant sub-group which contains $h_\lambda(n-1)^r$ and the Whitehead products $[\xi_\lambda, \eta]$, for every λ and every $\eta \in \pi_{r-n+2}(X)$.*

This will be proved in § 2 below.

Let $\{e_\lambda^n\}$ consist of a single cell e_1^n and let $\phi_1 | \dot{\sigma}^n$ be contractible in X. Then ϕ_1 can be extended to a map

$$\psi : (\dot{\sigma}^{n+1}, a_0 \dot{\sigma}^n) \to (Y, X), \tag{1.1}$$

where $\sigma^{n+1} = a_0 \sigma^n$ is an $(n+1)$-simplex, of which σ^n is a face. Let $f : n^r \to \pi_r(Y)$ be the homomorphism induced by ψ. Then $i_r^{-1}(0) = 0$, $f^{-1}(0) = 0$, and $i_r \pi_r(X) \cap f n^r = 0$ by Theorem 3 (a) in (**1**). Let $\alpha \in \pi_n(Y)$ be the element represented by $\psi : \dot{\sigma}^{n+1} \to Y$, let $q = r-n+1$, and let $[\alpha, i_q \pi_q(X)]$ be the group consisting of all the Whitehead products $[\alpha, i_q \beta]$ with $\beta \in \pi_q(X)$.

THEOREM 2. *Let $r < 2n$ and let $\pi_s(X) = 0$ if $1 \leqslant s \leqslant q-1 = r-n$. Then*†

$$\pi_r(Y) = i_r \pi_r(X) + fn^r + [\alpha, i_q \pi_q(X)].$$

An inductive argument, based on Theorem 2, leads to a similar theorem when $\{e^n_\lambda\}$ consists of more than one cell.‡ This more general theorem, applied to $\tilde{Y}$, leads to a similar theorem when‡ $\pi_1(X) \neq 1$, the remaining conditions of Theorem 2 being satisfied.

Let $P = X \cup X'$, where X, X' are closed, arcwise-connected sub-sets of the space P and let $X \cap X'$ be a single point p_0. Let

$$i: \pi_r(X) \to \pi_r(P), \qquad i': \pi_r(X') \to \pi_r(P)$$

be the injections and let

$$k: \pi_r(P) \to \pi_r(X), \qquad k': \pi_r(P) \to \pi_r(X')$$

be the left inverses of i, i' which are induced by the retractions

$$(P, X') \to (X, p_0), \qquad (P, X) \to (X', p_0).$$

Then [(**3**) § 8] $$\pi_r(P) = i\pi_r(X) + i'\pi_r(X') + \Gamma, \tag{1.2}$$

where§ $\Gamma = k^{-1}(0) \cap k'^{-1}(0)$. Let $Y = X \cup \{e^n_\lambda\}$ be the same as in Theorem 1, assuming that none of the cells e^n_λ meets X'. Let $Q = Y \cup X'$, with the topology which makes each of Y, X' closed sub-sets of Q. Let

$$j: \pi_r(Y) \to \pi_r(Q), \qquad j': \pi_r(X') \to \pi_r(Q)$$

be the injections and let $j'': \Gamma \to \pi_r(Q)$ be the homomorphism induced by the injection $\pi_r(P) \to \pi_r(Q)$.

THEOREM 3. *Let $\pi_s(X \cup X') = 0$ $(s = 1,..., r-n+1)$ and let $r < 2n-1$. Then*

$$\pi_r(Q) = j\pi_r(Y) + j'\pi_r(X') + j''\Gamma.$$

The rest of the paper consists of the proofs of these theorems. I wish to thank Professor J. H. C. Whitehead for his help in the preparation of the paper.

2. Proof of Theorem 1

Since any compact sub-set of Y is contained in the union of X and a finite sub-set of the cells $\{e^n_\lambda\}$, we may assume that the set of cells $\{e^n_\lambda\}$ is finite. Let

$$Y = X \cup e^n_1 \cup ... \cup e^n_m$$

† $G = G_1 + G_2 + ...$ will always mean that the (additive, Abelian) group G is the direct sum of its sub-groups G_1, G_2,... .

‡ Cf. the reduction of Theorem 1, in § 2 below, to the case in which $Y = X \cup e^n$, $\pi_1(Y) = 1$.

§ $\Gamma \approx \pi_{r+1}(X \times X', P)$, when the points (x, p_0) and (p_0, x') in $X \times X'$ are identified with $x \in X$ and $x' \in X'$ [cf. (**6**)].

and assume that the theorem is true if $m = 1$. If $m > 1$, let $X' = Y - e^n_m$ and let

$$i' : \pi_r(X) \to \pi_r(X'), \qquad i'' : \pi_r(X') \to \pi_r(Y)$$

be the injections. Then $i_r = i''i'$ and h_m, ξ_m become $i'h_m$, $i'\xi_m$ when X is replaced by X'. Let $\alpha \in i_r^{-1}(0)$. Then $i''i'\alpha = 0$, and it follows from our assumption that

$$i'\alpha = i'h_m \beta + [i'\xi_m, \eta'],$$

where $\beta \in (n-1)^r$ and $\eta' \in \pi_{r-n+2}(X')$. Since $r-n+2 < n$, it follows that $\eta' = i'\eta$, for some $\eta \in \pi_{r-n+2}(X)$. Therefore

$$i'(\alpha - h_m \beta - [\xi_m, \eta]) = 0$$

and the theorem follows by induction on m. So we assume that $Y = X \cup e^n$ and discard the subscript λ in h_λ, ξ_λ, etc.

If $r < n-1$, the theorem is trivial, and it is known† if $r = n-1$. Therefore we assume that $r \geqslant n$, in which case $n > 2$ since $r < 2n-2$. Then $\pi_1(X) \approx \pi_1(Y)$. If $\pi_1(Y) \neq 1$, we replace e^n by the interior e^n_0 of an n-element $E^n \subset e^n$, and X by $X_0 = Y - e^n_0$, of which X is a deformation retract.‡ Let $\tilde{Y}$ be a universal covering space of Y. Then

$$\tilde{Y} = \tilde{X}_0 \cup \{\tilde{e}^n_\lambda\},$$

where $\tilde{X}_0$ covers X_0 and each of the cells $\tilde{e}^n_\lambda$ covers e^n_0. It follows from an argument on p. 232 of (**5**) that no compact sub-set of $\tilde{Y}$ meets infinitely many of the cells $\tilde{e}^n_\lambda$. Therefore we may proceed as before, with Y replaced by $\tilde{Y}$. Therefore we assume from the first that $Y = X \cup e^n$, $r \geqslant n > 2$ and that $\pi_1(X) = 1$.

Let e^n have the Euclidean geometry of a convex n-cell and let σ^n_0 be a rectilinear simplex in e^n. Let $u_0 : \dot{\sigma}^{r+1} \to X$ be a map which represents a given element $\alpha \in i_r^{-1}(0)$, where σ^{r+1} is a rectilinear $(r+1)$-simplex. Then u_0 has an extension $v : \sigma^{r+1} \to Y$. We assume that $v^{-1}\sigma^n_0$ is a simplicial complex, with respect to which $v | v^{-1}\sigma^n_0$ is a simplicial map onto§ σ^n_0. Let y_0 be an interior point of σ^n_0. Then $v^{-1}y_0$ is an $(m-1)$-dimensional polyhedron P^{m-1}, where $m = r-n+2$. Let $p_0 \in \sigma^{r+1} - \dot{\sigma}^{r+1}$ be a point which is in general position relative to P^{m-1}. Since

$$2m-1 = 2r-2n+3 < r+1,$$

it follows that, as the point p describes P^{m-1}, the segment $p_0 p$ sweeps out a cone C^m which is a non-singular model of the join $p_0 P^{m-1}$.

† Cf. Theorem 18 in (**4**).

‡ We can take $E^n = \phi\sigma^n_1$, where $\phi : \sigma^n \to \bar{e}^n$ means the same as in § 1, and σ^n_1 is a rectilinear simplex in the interior of σ^n.

§ Cf. the first part of the proof of Lemma 1 in (**1**). We can always deform v into a map which covers σ^n_0 though, obviously, $\alpha = 0$ if $v\sigma^{r+1} \subset Y - y_0$.

Let E^{r+1} be a rectilinear triangulation of σ^{r+1}, sub-complexes of which cover $v^{-1}\sigma_0^n$, C^m, and P^{m-1}. Let E_2^{r+1} be the second derived complex of E^{r+1} and let C_2^m be the sub-complex of E_2^{r+1}, which covers C^m. Let $N^{r+1} \subset E_2^{r+1}$ be the sub-complex consisting of all the (closed) simplexes which meet P^{m-1} and let $D^m = N^{r+1} \cap C_2^m$. Then D^m is the (closed) simplicial neighbourhood of P^{m-1} in C_2^m. Since $v^{-1}\sigma_0^n$ is covered by a sub-complex of E^{r+1}, we have $N^{r+1} \subset v^{-1}\sigma_0^n$, whence $vN^{r+1} \subset \sigma_0^n$. It follows from the classical theory of simplicial neighbourhoods† that there is a retracting deformation of N^{r+1} onto P^{m-1}, which deforms D^m over itself. Therefore D^m is a deformation retract of N^{r+1}: that is to say, there is a homotopy $\rho_t: N^{r+1} \to N^{r+1}$ such that $\rho_0 = 1$, $\rho_1 N^{r+1} = D^m$, and‡ $\rho_t p = p$ if $p \in D^m$. This may be extended to a retracting deformation (rel C^m) of $C^m \cup N^{r+1}$ onto C^m. Therefore C^m is a deformation retract of $C^m \cup N^{r+1}$. Since C^m, being a cone, is contractible, so is $C^m \cup N^{r+1}$. Therefore $C^m \cup N^{r+1}$ is a deformation retract of E^{r+1}.

Since $r+1-(m-1) = n > 2$, it follows that a singular 2-cell in E^{r+1} can be freed from P^{m-1} by a small deformation near P^{m-1}. Therefore

$$\pi_1(E^{r+1}-P^{m-1}) = 1, \qquad \pi_1\{(C^m \cup N^{r+1})-P^{m-1}\} = 1.$$

Moreover $N^{r+1}-P^{m-1}$ can be projected outwards onto $\dot{N}^{r+1}$. Hence it follows that

$$\pi_1(E^{r+1}-U^{r+1}) = 1, \qquad \pi_1(K) = 1, \tag{2.1}$$

where $U^{r+1} = N^{r+1}-\dot{N}^{r+1}$, $K = (C_2^m \cup N^{r+1})-U^{r+1}$. Since

$$H_p(E^{r+1}, C^m \cup N^{r+1}) = 0 \quad (p = 0, 1,...),$$

where H_p indicates a pth relative homology group, it follows from the strong form of the excision theorem,§ which applies to polyhedra, that

$$H_p(E^{r+1}-U^{r+1}, K) = 0.$$

Hence it follows from (2.1) that

$$\pi_p(E^{r+1}-U^{r+1}, K) = 0 \quad (p = 2, 3,...).$$

Thus K is a deformation retract of $E^{r+1}-U^{r+1}$. Therefore the identical map $\dot{\sigma}^{r+1} \to \sigma^{r+1}$ is homotopic in $E^{r+1}-U^{r+1}$ to a map $\theta: \dot{\sigma}^{r+1} \to K$.

Since $E^{r+1}-U^{r+1}$ is compact, there is a rectilinear simplex $\sigma_1^n \subset \sigma_0^n$ which contains y_0 in its interior e_1^n and is contained in vU^{r+1}. Let

† See (**7**), 91.

‡ See (**8**) and (**9**).

§ That is $i: H_n(X-U, A-U) \approx H_n(X, A)$, where i is the injection and U is a sub-set of A, which is open in X. A weaker form of this is stated as Axiom 6 in (**10**).

$X_1 = Y - e_1^n$ and let X, e^n be replaced by X_1, e_1^n. Let $\rho_t' : \sigma_0^n \to \sigma_0^n$ be a homotopy ($\mathrm{rel}\,\sigma_1^n$) in which $\sigma_0^n - \sigma_1^n$ collapses onto $\dot{\sigma}_1^n$ and let

$$\rho_t'' : (Y, X_1) \to (Y, X_1)$$

be an extension of ρ_t'. Let $w_0 = \rho_1'' v | K$. Then $w_0 \dot{N}^{r+1} \subset \dot{\sigma}_1^n$, since $v\dot{N}^{r+1} \subset \sigma_0^n - \sigma_1^n$, and w_0 has an extension $K \cup U^{r+1} \to Y$, namely $\rho_1'' v | K \cup U^{r+1}$, which maps N^{r+1} into σ_1^n. Let $L \subset K$ be the sub-complex which covers the closure of $C^m - D^m$ and let L^{m-1} be its $(m-1)$-section. Since $N^{r+1} \cap C^m = D^m$, which is the simplicial neighbourhood of P^{m-1} in C_2^m, it follows that $C^m \cap \dot{N}^{r+1} \subset L^{m-1}$. Since $\pi_s(X_1) = 0$ for $s \leqslant m-1$ and since $m-1 = r-n+1 < n-1$, it follows that there is a homotopy†

$$w_t : (K, \dot{N}^{r+1}) \to (X_1, \dot{\sigma}_1^n)$$

such that $w_1 L^{m-1} = y_1$, where $y_1 \in \dot{\sigma}_1^n$. Therefore the original map $u_0 : \dot{\sigma}^{r+1} \to X$ is homotopic in X_1 to

$$u_1 = w_1 \theta : \dot{\sigma}^{r+1} \to X_1. \tag{2.2}$$

Moreover $w_1 | \dot{N}^{r+1}$ has an extension $N^{r+1} \to \sigma_1^n$.

Let $\sigma_1^m, \ldots, \sigma_t^m$ be the m-simplexes in L and let $S_1^m, \ldots, S_t^m$ be a set of m-spheres which are disjoint from σ_1^n and from each other. Let $q_i \in S_i^m$ and let $q_1, \ldots, q_t$ be identified with y_1 so as to form a polyhedron

$$Q = S^{n-1} \cup \Sigma^m \quad (S^{n-1} = \dot{\sigma}_1^n),$$

where Σ^m is the union of the m-spheres, which we still denote by $S_1^m, \ldots, S_t^m$. Thus

$$S_i^m \cap S_j^m = \Sigma^m \cap \sigma_1^n = y_1 \quad (i \neq j),$$

and it follows that S^{n-1}, Σ^m are retracts of Q. Therefore, if $T = S^{n-1}$ or Σ^m, the injection $i : \pi_r(T) \to \pi_r(Q)$, is an isomorphism into $\pi_r(Q)$, and we identify each $\tau \in \pi_r(T)$ with $i\tau$. Let $\lambda \in \pi_r(S^{n-1})$, $\mu_i \in \pi_m(Q)$ be the elements represented by the identical maps of S^{n-1}, S_i^m, when the latter are oriented, and let Λ be the sub-group of $\pi_r(Q)$ which is generated by the products $[\lambda, \mu_i]$. It follows from the proof of Theorem 2 in (**2**) that

$$\pi_r(Q) = \pi_r(S^{n-1}) + \pi_r(\Sigma^m) + \Lambda, \tag{2.3}$$

and that Λ is freely generated, with commutative addition, by the elements $[\lambda, \mu_i]$.

Let‡ $s_i : (\sigma_i^m, \dot{\sigma}_i^m) \to (S_i^m, y_1)$ be a map such that $s_i | \sigma_i^m - \dot{\sigma}_i^m$ is a homeomorphism onto $S_i^m - y_1$. Let $w' : K \to Q$ be the map which is given by $w'p = w_1 p$ or $s_i p$ according as $p \in \dot{N}^{r+1}$ or $p \in \sigma_i^m$. It is single-valued, and hence obviously continuous, because $L \cap \dot{N}^{r+1} \subset L^{m-1}$. For the

† Cf. the proof of Lemma 1 in (**1**).

‡ The rest of the proof is similar to the last part of the proof of Lemma 4 in (**2**).

same reason, and, since $w' = w_1$ in $\dot{N}^{r+1}$ and $w'|\sigma_i^m - \dot{\sigma}_i^m$ is a homeomorphism onto $S_i^m - y_1$, a map $w'': Q \to X_1$ is defined by $w''w' = w_1$ ($w''q = q$ if $q \in S^{n-1}$). Since $w_1|\dot{N}^{r+1}$ has an extension $N^{r+1} \to \sigma_1^n$, it follows that w' has an extension $w^*: K \cup N^{r+1} \to Q \cup \sigma_1^n$. Let

$$j: \pi_r(K) \to \pi_r(K \cup N^{r+1}), \qquad k: \pi_r(Q) \to \pi_r(Q \cup \sigma_1^n)$$

be the injections and let

$$g': \pi_r(K) \to \pi_r(Q), \qquad g^*: \pi_r(K \cup N^{r+1}) \to \pi_r(Q \cup \sigma_1^n)$$

be the homomorphisms induced by w' and w^*. Obviously $\pi_r(Q \cup \sigma_1^n)$ may be identified with the summand $\pi_r(\Sigma^m) \subset \pi_r(Q)$ in such a way that k becomes the homomorphism induced by the retraction

$$(Q, S^{n-1}) \to (\Sigma^m, y_1).$$

Then it follows from (2.3) that

$$k^{-1}(0) = \pi_r(S^{n-1}) + \Lambda.$$

Since $\pi_r(K \cup N^{r+1}) = 0$, because $K \cup N^{r+1}$ is contractible, and since $kg' = g^*j$, it follows that

$$g'\pi_r(K) \subset \pi_r(S^{n-1}) + \Lambda.$$

Since $w_1 = w''w'$, the theorem now follows from (2.2).

3. Proof of Theorem 2

The theorem is trivial or known† if $r \leqslant n$. Therefore we assume that $r > n$. As in the proof of Theorem 3 in (**1**) we may assume that $Y = X \cup S^n$ and $\psi(a_0 \dot{\sigma}^n) = x_0$, where $S^n = \bar{e}_1^n$ is an n-sphere which meets X in a single point x_0, and ψ means the same as in (1.1). Let $S^n = \dot{E}^{n+1}$, where E^{n+1} is an n-element such that $X \cap E^{n+1} = x_0$, and let

$$Z = X \cup E^{n+1} = Y \cup e^{n+1} \quad (e^{n+1} = E^{n+1} - S^n).$$

Let $j: \pi_r(Y) \to \pi_r(Z)$ be the injection. Obviously $\pi_r(Z)$ may be identified with $\pi_r(X)$ in such a way that j becomes a left inverse of i_r. Therefore

$$\pi_r(Y) = i_r \pi_r(X) + j^{-1}(0). \tag{3.1}$$

Since $r-(n+1)+1 = r-n$, $2(n+1)-2 = 2n$, the conditions of Theorem 1 are satisfied when X, Y, n are replaced by $Y, Z, n+1$, Also h_1, in Theorem 1, becomes f. Therefore $j^{-1}(0)$ is generated by the elements in the set-theoretic union of fn^r and $[\alpha, i_q \pi_q(X)] = \Pi$, say. Let $l: \pi_r(Y) \to n^r$ be the homomorphism induced by the retraction $(Y, X) \to (S^n, x_0)$. Obviously $lf = 1: n^r \to n^r$ and $l\Pi = 0$, whence $fn^r \cap \Pi = 0$. Therefore $j^{-1}(0) = fn^r + \Pi$ and the theorem follows from (3.1).

† Cf. Theorem 19 in (**4**).

4. Proof of Theorem 3

We recall that

$$Y = X \cup \{e_\lambda^n\}, \qquad Q = Y \cup X' = P \cup \{e_\lambda^n\}, \qquad Y \cap X' = p_0,$$

and we have to prove that

$$\pi_r(Q) = j\pi_r(Y) + j'\pi_r(X') + j''\Gamma.$$

Since $\Gamma = k^{-1}(0) \cap k'^{-1}(0)$, it follows that $j''\Gamma \to 0$ in the homomorphism induced by each of the retractions $(Q, Y) \to (X', p_0)$ and $(Q, X') \to (Y, p_0)$. Therefore it follows from an argument similar to the one at the end of § 3 that, if $a'+b+c = 0$, where $a' \in j'\pi_r(X')$, $b \in j\pi_r(Y)$, $c \in j''\Gamma$, then $a' = b = c = 0$. Therefore it only remains to prove that the combined elements in the groups $j'\pi_r(X')$, $j\pi_r(Y)$, $j''\Gamma$ generate $\pi_r(Q)$. It follows from (1.2) that this is equivalent to proving that $\pi_r(Q)$ is generated by the combined elements in $j\pi_r(Y)$ and $h\pi_r(P)$, where $h: \pi_r(P) \to \pi_r(Q)$ is the injection.

As in the proof of Theorem 1 we may assume that

$$Y = X \cup e_1^n \cup \ldots \cup e_m^n$$

for some finite value of m. Let $m > 1$ and assume that Theorem 3 is true if $m = 1$. Let $\alpha \in \pi_r(Q)$ be given and let $X^* = Y - e_m^n$. Then it follows from the theorem, with $m = 1$ and X replaced by X^*, that $\alpha = h^*\alpha^* + j\eta$, where

$$\alpha^* \in \pi_r(X^* \cup X'), \qquad \eta \in \pi_r(Y)$$

and $h^*: \pi_r(X^* \cup X') \to \pi_r(Q)$ is the injection. Now replace Y by X^*, let $Q^* = X^* \cup X'$ and assume that $\alpha^* = h_1^*\xi + j^*\eta^*$, where $\xi \in \pi_r(P)$, $\eta^* \in \pi_r(X^*)$ and

$$h_1^*: \pi_r(P) \to \pi_r(Q^*), \qquad j^*: \pi_r(X^*) \to \pi_r(Q^*)$$

are the injections. Obviously $h^*h_1^* = h$ and $h^*j^* = ji^*$, where

$$i^*: \pi_r(X^*) \to \pi_r(Y)$$

is the injection. Therefore $\alpha = h\xi + j\eta_1$, where

$$\eta_1 = i^*\eta^* + \eta \in \pi_r(Y).$$

Therefore the theorem follows by induction on m, assuming that it is true when $m = 1$.

Let $Y = X \cup e^n$ and let $v_0: S^r \to Q$ be a map which represents a given element $\alpha \in \pi_r(Q)$. Since $\pi_s(P) = 0$ for $s = 1,\ldots, r-n+1$ and $r < 2n-1$, we may assume, in consequence of Lemma 1 in (1), that $v_0^{-1}y_0$ is a single point $q_0 \in S^r$, where y_0 is a point in e^n. Let E_0^r and E_1^r be 'northern' and 'southern' hemispheres on S^r, the point q_0 being the 'north pole',

and let $S^{r-1} = \dot{E}^r_i$ ($i = 0, 1$). We may assume that $v_0 E^r_0 \subset \bar{e}^n$ and $v_0 E^r_1 \subset P$. Then $v_0 S^{r-1} \subset X$ and $v_0|S^{r-1}$ is contractible in P. Since X is a retract of P, because $X \cap X' = p_0$, and since X is arcwise-connected, it follows that $v_0|S^{r-1}$ is homotopic in X to the constant map $S^{r-1} \to p_0$. Consequently it follows from the homotopy extension-theorem, applied to $v_0|E^r_0$ in Y and to $v_0|E^r_1$ in P, that there is a homotopy, $v_t: S^r \to Q$, such that $v_1 E^r_0 \subset Y$, $v_1 E^r_1 \subset P$, and $v_1 S^{r-1} = p_0$. Therefore $\alpha = h\xi + j\eta$, where $\eta \in \pi_r(Y)$ and $\xi \in \pi_r(P)$ are represented by $v_1|E^r_0$ and $v_1|E^r_1$. This proves the theorem.

REFERENCES

1. J. H. C. Whitehead, 'Note on suspension', *Quart. J. of Math.* (Oxford), (2) *supra*, 9–22.
2. —— 'On adding relations to homotopy groups', *Annals of Math.* 42 (1941), 409–28.
3. —— 'The homotopy type of a special kind of polyhedron', *Annales Soc. Polonaise Math.* 21 (1948), 176–86.
4. —— 'Simplicial spaces, nuclei, and *m*-groups', *Proc. London Math. Soc.* 45 (1939), 243–327.
5. —— 'Combinatorial homotopy (I)', *Bull. American Math. Soc.* 55 (1949), 213–45.
6. G. W. Whitehead, 'A generalization of the Hopf invariant', *Proc. National Acad. of Sci.* 32 (1946), 188–90.
7. S. Lefschetz, *Topology* (New York, 1930).
8. R. H. Fox, 'On homotopy type and deformation retracts', *Annals of Math.* 44 (1943), 40–50.
9. H. Samelson, 'Remark on a paper by R. H. Fox', ibid. 45 (1944), 448–9.
10. S. Eilenberg and N. E. Steenrod, 'Axiomatic approach to homology theory', *Proc. National Acad. of Sci.* 31 (1945), 117–20.

ON THE ANALOGUE OF DIXON'S THEOREM FOR BILATERAL BASIC HYPERGEOMETRIC SERIES

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[Received 5 December 1949]

1. A FORMULA equivalent to

$$ {}_3\Phi_2\left[\begin{matrix} a, & b, & c; \\ & aq/b, & aq/c \end{matrix} q^2\sqrt{a}/bc\right] = \frac{(q)_{2n}(b)_n(c)_n(bc)_{2n}}{(q)_n(b)_{2n}(c)_{2n}(bc)_n}, \tag{1.1} $$

where $a = q^{-2n}$ and n is a positive integer, was given by F. H. Jackson.* I use the notation

$$ (a)_n = (1-a)(1-aq)...(1-aq^n), $$
$$ (a)_{-n} = 1/\{(1-a/q)(1-a/q^2)...(1-a/q^n)\}. $$

The main interest of this result is that it gives a direct analogue of Dixon's theorem in the case when a has the above special form, though no such analogue exists for general values of a.

Now, if we write the series on the left of (1.1) as $\sum_{r=0}^{2n}$, put $r = n+s$, and write b, c, d for bq^n, cq^n, q^{-n}, the formula becomes

$$ {}_3\Psi_3\left[\begin{matrix} b, & c, & d; \\ q/b, & q/c, & q/d \end{matrix} q^2/bcd\right] = \Pi\left[\begin{matrix} q, & q/bc, & q/bd, & q/cd; \\ q/b, & q/c, & q/d, & q/bcd \end{matrix}\right], \tag{1.2} $$

where

$$ \Pi(a) = \prod_{n=0}^{\infty}(1-aq^n), \qquad \Pi\left[\begin{matrix} a_1, a_2,..., a_r; \\ b_1, b_2,..., b_s \end{matrix}\right] = \frac{\Pi(a_1)\Pi(a_2)...\Pi(a_r)}{\Pi(b_1)\Pi(b_2)...\Pi(b_s)}, $$

provided that one of b, c, d is of the form q^{-n}, so that the series terminates both above and below. We shall see that the result is still true when the series does not terminate.

The formula (1.1) and some similar results have so far seemed to be rather isolated results, but I now find that, when written in terms of bilateral series, they can be obtained very easily from known transformations of unilateral basic series.

2. We begin with the formula

$$ {}_6\Phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d; \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d \end{matrix} aq/bcd\right] $$
$$ = \Pi\left[\begin{matrix} aq, & aq/bc, & aq/bd, & aq/cd; \\ aq/b, & aq/c, & aq/d, & aq/bcd \end{matrix}\right], \tag{2.1} $$

* See Jackson (3) and Bailey (1).

which is a limiting case of Jackson's analogue of Dougall's theorem.* In this we let $a \to 1$, and we get on the left

$$1+\sum_{n=1}^{\infty}\frac{(b)_n(c)_n(d)_n}{(q/b)_n(q/c)_n(q/d)_n}(1+q^n)(q/bcd)^n.$$

Now $$\frac{(b)_{-n}(c)_{-n}(d)_{-n}}{(q/b)_{-n}(q/c)_{-n}(q/d)_{-n}}\left(\frac{q}{bcd}\right)^{-n}=\frac{(b)_n(c)_n(d)_n}{(q/b)_n(q/c)_n(q/d)_n}\left(\frac{q^2}{bcd}\right)^n,$$

and so (2.1) becomes

$$_3\Psi_3\left[\begin{matrix}b, & c, & d; \\ q/b, & q/c, & q/d\end{matrix}\; q/bcd\right]=\Pi\left[\begin{matrix}q, & q/bc, & q/bd, & q/cd; \\ q/b, & q/c, & q/d, & q/bcd\end{matrix}\right], \tag{2.2}$$

where the series $_3\Psi_3$ need not terminate. It will be noticed that the formula remains true when the argument q/bcd is changed into q^2/bcd, the series so obtained being simply the above series written in the reverse order.

Similarly, if we take $a = q$ in (2.1) and multiply by $1-q$, we get

$$_3\Psi_3\left[\begin{matrix}b, & c, & d; \\ q^2/b, & q^2/c, & q^2/d\end{matrix}\; q^2/bcd\right]=\Pi\left[\begin{matrix}q, & q^2/bc, & q^2/bd, & q^2/cd; \\ q^2/b, & q^2/c, & q^2/d, & q^2/bcd\end{matrix}\right]. \tag{2.3}$$

Now it is evident that

$$_3\Psi_3\left[\begin{matrix}b, & c, & d; \\ e, & f, & g\end{matrix}\; x\right]=\sum_{n=-\infty}^{\infty}\frac{(b)_{n+r}(c)_{n+r}(d)_{n+r}}{(e)_{n+r}(f)_{n+r}(g)_{n+r}}x^{n+r}$$

$$=\frac{(b)_r(c)_r(d)_r}{(e)_r(f)_r(g)_r}x^r\,{}_3\Psi_3\left[\begin{matrix}bq^r, & cq^r, & dq^r; \\ eq^r, & fq^r, & gq^r\end{matrix}\; x\right].$$

This result, together with (2.2) and (2.3), shows that *there is a direct analogue of Dixon's theorem for bilateral basic series when the common product of pairs of parameters is an integral power of q.*

3. Similar results can be written down from other known formulae for unilateral basic series. For example, from Jackson's analogue of Dougall's theorem we get

$$_5\Psi_5\left[\begin{matrix}b, & c, & d, & e, & q^{-n}; \\ q/b, & q/c, & q/d, & q/e, & q^{n+1}\end{matrix}\; q\right]=\frac{(q)_n(q/cd)_n(q/bd)_n(q/bc)_n}{(q/b)_n(q/c)_n(q/d)_n(q/bcd)_n}, \tag{3.1}$$

where $bcde = q^{n+1}$, and

$$_5\Psi_5\left[\begin{matrix}b, & c, & d, & e, & q^{-n}; \\ q^2/b, & q^2/c, & q^2/d, & q^2/e, & q^{n+2}\end{matrix}\; q\right]=\frac{(q)_{n+1}(q^2/cd)_n(q^2/bd)_n(q^2/bc)_n}{(q^2/b)_n(q^2/c)_n(q^2/d)_n(q^2/bcd)_n}, \tag{3.2}$$

* Jackson's identity is given in (**2**), § 8.3 (1). The formula (2.1) is obtained by substituting for e and letting $N \to \infty$.

where $bcde = q^{n+3}$. The formula (3.1) is equivalent to the most general formula given by Jackson in the paper referred to above.

We can, of course, write down similar results giving transformations of bilateral series of the type considered above. Those obtained from Watson's transformation of a well-poised ${}_8\Phi_7$ into a Saalschützian ${}_4\Phi_3$ make it particularly obvious why, in obtaining the Rogers–Ramanujan identities, the limiting form of Watson's transformation, when five parameters tend to infinity, gives a series on the left which can be summed by Jacobi's theorem when $a = 1$ and when $a = q$.

It is perhaps worth noting that if, in (2.2) and (2.3), we take

$$b = c = d = q^{-n},$$

we obtain the formulae

$$\sum_{r=-n}^{n} (-1)^r \binom{2n}{n+r}^3 q^{\frac{1}{2}r(3r+1)} = \frac{(q)_{3n}}{\{(q)_n\}^3}, \tag{3.3}$$

$$\sum_{r=-n-1}^{n} (-1)^r \binom{2n+1}{n+r+1}^3 q^{\frac{1}{2}r(3r+1)} = \frac{(q)_{3n+1}}{\{(q)_n\}^3}, \tag{3.4}$$

where
$$\binom{n}{r} = \frac{(q)_n}{(q)_r (q)_{n-r}}.$$

These formulae correspond to those giving the sums of the cubes of the coefficients in the binomial expansion of $(1-x)^m$ in the cases when the index m is even or odd; (3.3) is equivalent to the final formula in (1).

REFERENCES

1. W. N. Bailey, *Quart. J. of Math.* (Oxford), 12 (1941), 173–5.
2. —— *Generalized hypergeometric series* (Cambridge, 1935).
3. F. H. Jackson, *Quart. J. of Math.* (Oxford), 12 (1941), 167–72.